UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 26 May, 2016 1:30 pm to 4:30 pm

PAPER 213

STOCHASTIC NETWORKS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Define an *open migration process.* Establish the form of the equilibrium distribution, giving conditions for its existence.

In a stationary open migration process, each colony j acts as a single-server queue at which service times are exponentially distributed with parameter ϕ_j. Let α_j be the mean arrival rate at queue j. If $\phi_j, j = 1, 2, \ldots, J$, can be chosen subject to the constraint

$$\sum_{j=1}^{J} \phi_j = F,$$

find the choice, in terms of F and α_j, $j = 1, 2, \ldots, J$, that minimizes the mean number of individuals in the system.

(ii) Write an essay on the concept of an *effective bandwidth.* Your essay should include a derivation of Chernoff's bound, and an interpretation of the expression

$$\alpha(s) = \frac{1}{s} \log \mathbb{E}[e^{sX}]$$

when the parameter s approaches zero and infinity.

2

Define *a loss network with fixed routing.* Show that its transition rates can be written in the form

$$q(n, n + e_r) = \nu_r, \quad n + e_r \in S(C)$$
$$q(n, n - e_r) = n_r, \quad n - e_r \in S(C)$$

and its stationary distribution in the form

$$\pi(n) = G(C) \prod_r \frac{\nu_r^{n_r}}{n_r!}, \quad n \in S(C)$$

where

$$S(C) = \{n : An \leqslant C\}.$$

Give an interpretation of the vectors n, C and e_r, and of the matrix A. What is the interpretation of the vector $m = C - An$?

Let

$$\pi'(m) = \sum_{n : An = C - m} \pi(n).$$

By calculating $\mathbb{E}[n_r \mid m]$, or otherwise, prove that

$$(C_j - m_j)\, \pi'(m) = \sum_{r : Ae_r \leqslant C - m} A_{jr} \nu_r \pi'(m + Ae_r).$$

3

A finite set, R, of stations attempt to share a channel. The stations form the vertex set of an interference graph, with an edge between two stations if they interfere with each other. Let $S \subset \{0,1\}^R$ be the set of vectors with the property that if stations i and j have an edge between them then $n_i.n_j = 0$. Thus if $n \in S$ the stations r with $n_r = 1$ can transmit simultaneously without interference. Describe a Markov process, representing a simple random access scheme, with stationary distribution

$$\pi_\theta(n) = \frac{\exp(n.\theta)}{\sum_{m \in S} \exp(m.\theta)}$$

where $\theta = (\theta_r, r \in R)$.

Let $\lambda = (\lambda_r, r \in R)$ and

$$\Lambda = \{\lambda \geqslant 0 : \exists p(n) \geqslant 0, \sum_{n \in S} p(n) = 1 \text{ such that } \sum_{n \in S} p(n) n_r \geqslant \lambda_r \text{ for } r \in R\}.$$

Prove that, provided λ lies in the interior of the region Λ, there exists a vector θ such that

$$\mathbb{E}_\theta[n_r] > \lambda_r, \quad r \in R$$

where $\mathbb{E}_\theta[n_r] = \sum_{n \in S} n_r \pi_\theta(n)$. Interpret this result, in terms of the throughputs that can be achieved by the random access scheme.

4

The dynamical system

$$\begin{aligned} \frac{d}{dt}x_r(t) &= \kappa_r \left(w_r - x_r(t) \textstyle\sum_{j \in r} \mu_j(t) \right) & r \in R \\ \mu_j(t) &= p_j \left(\textstyle\sum_{s: j \in s} x_s(t) \right) & j \in J \end{aligned}$$

is proposed as a model for a communication network, where R is a set of routes, J is a set of resources, $\kappa_r > 0$ and $w_r > 0$, $r \in R$, and $p_j(\cdot)$, $j \in J$, are non-negative, continuous, increasing functions.

Provide a brief interpretation of this model, in terms of feedback signals generated by resources and acted upon by users. In what sense does the system use only local information?

By considering the function

$$U(x) = \sum_{r \in R} w_r \log x_r - \sum_{j \in J} \int_0^{\sum_{s: j \in s} x_s} p_j(y) dy$$

or otherwise, show that all trajectories of the dynamical system converge towards a unique equilibrium point.

An alternative dynamical system

$$\frac{d}{dt}\mu_j(t) = \kappa_j \, \mu_j(t) \left(\sum_{r: j \in r} x_r(t) - q_j\left(\mu_j(t)\right) \right) \qquad j \in J$$

$$x_r(t) = \frac{w_r}{\sum_{k \in r} \mu_k(t)} \qquad r \in R$$

is proposed, where, for $j \in J$, $\kappa_j > 0$, and $q_j(\cdot)$ is a continuous, strictly increasing function with $q_j(0) = 0$. Show that this dynamical system has a unique equilibrium point.

Discuss briefly any possible relationship between the equilibrium points of the two dynamical systems.

5

Let J be a set of resources, and R a set of routes, where a route $r \in R$ identifies a subset of J. Let C_j be the capacity of resource j, and suppose the number of flows in progress on each route is given by the vector $n = (n_r, r \in R)$. A rate allocation $x = (x_r, r \in R)$ is *feasible* if

$$x_r \geqslant 0, \; r \in R, \text{ and } \sum_{r:j \in r} n_r x_r \leqslant C_j \quad j \in J.$$

A rate allocation is *proportionally fair* if it is feasible and if, for any other feasible rate allocation $y = (y_r, r \in R)$,

$$\sum_{r \in R} n_r \left(\frac{y_r - x_r}{x_r} \right) \leqslant 0.$$

Show that a proportionally fair rate allocation solves the optimization problem

$$\begin{aligned} \text{maximize} \quad & \sum_r n_r \log x_r \\ \text{subject to} \quad & \sum_{r:j \in r} n_r x_r \leqslant C_j \quad j \in J \\ \text{over} \quad & x_r \geqslant 0, r \in R. \end{aligned}$$

Consider a *linear* network with resources $J = \{1, 2, \ldots, I\}$, each of unit capacity, and routes $R = \{0, 1, 2, \ldots, I\}$ where we use the symbol 0 to represent a route $\{1, 2, \ldots, I\}$ which traverses the entire set of resources, and the symbol i to represent a route $\{i\}$ through a single resource, for $i = 1, 2, \ldots, I$. Show that under a proportionally fair rate allocation

$$x_0 n_0 + x_i n_i = 1 \quad \text{if} \quad n_i > 0, \quad i = 1, 2, \ldots, I$$

and

$$x_0 = \frac{1}{n_0 + \sum_{i=1}^{I} n_i} \quad \text{if} \quad n_0 > 0.$$

Suppose now that flows describe the transfer of documents through the linear network above, that new flows originate as independent Poisson processes of rates $\nu_r, r \in R$, and that document sizes are independent and exponentially distributed with parameter μ_r for each route $r \in R$. Determine the transition intensities of the resulting Markov process $n = (n_r, r \in R)$. Show that the stationary distribution of the Markov process $n = (n_r, r \in R)$ takes the form

$$\pi(n) = (1 - \rho_0)^{1-I} \prod_{i=1}^{I} (1 - \rho_0 - \rho_i) \binom{\sum_{r=0}^{I} n_r}{n_0} \prod_{r=0}^{I} \rho_r^{n_r},$$

provided $\rho_0 + \rho_i < 1$, $i = 1, 2, \ldots, I$, where $\rho_r = \nu_r / \mu_r$. Show that, under the distribution π,

$$\mathbb{E}(n_i) = \frac{\rho_i}{1 - \rho_0 - \rho_i} \qquad i = 1, 2, \ldots, I.$$

END OF PAPER