UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 31 May, 2016 9:00 am to 12:00 pm

# PAPER 212

# MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Use the simplex algorithm to solve the problem

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 3x_2 \\ 2x_1 + x_2 \quad & \leqslant 11 \\ -x_1 + 2x_2 \quad & \leqslant 6 \\ x_1, x_2 \quad & \geqslant 0. \end{aligned}$$

Use Gomory cuts and the dual simplex algorithm to solve the problem under the additional constraint that $x_1$ and $x_2$ are integers. Your first cut can be obtained from either the first or second row of the simplex tableau. Determine, and use, the choice that cuts most from the feasible region.

**2**

Consider the following two problems.

**Subset sum**. Given nonnegative $s_1, \dots, s_n$ and $t$, does there exist a subset of the $s_1, \dots, s_n$ whose sum is $t$?

**Knapsack optimization**. Given $v_1, \dots, v_n$, $w_1, \dots, w_n$ and $B$, maximize $\sum_i x_i v_i$ subject to $\sum_i x_i w_i \leqslant B$ and $x_1, \dots, x_n \in \{0, 1\}$.

Define the problem classes of NP, NP-complete and NP-hard. Assuming that Subset sum is NP-complete, prove that Knapsack optimization is NP-hard.

Show that there exists a 1/2–approximation algorithm for Knapsack optimization.

In the Quadratic knapsack optimization problem (QKP) one is also given a symmetric matrix of elements $p_{ij}$ and is to maximize $\sum_i x_i v_i + \sum_i \sum_{j \neq i} p_{ij} x_i x_j$, subject to $\sum_i x_i w_i \leqslant B$ and $x_1, \dots, x_n \in \{0, 1\}$. Show that

$$\sum_{j \neq i} x_i x_j w_j \leqslant (B - w_i) x_i.$$

Let

$$q_i = \max \left\{ \sum_{j \neq i} y_j p_{ij} \ : \ \sum_{j \neq i} y_j w_j \leqslant B - w_i,\ 0 \leqslant y_j \leqslant 1 \right\}.$$

Show that an upper bound for QKP can be found in polynomial time by computing $q_1, \dots, q_n$ and then solving the Lagrangian relaxation of one further knapsack problem.

Suppose $n = 3$, $B = 4$, $w = (1, 2, 3)$, $v = (10, 20, 15)$,

$$(p_{ij}) = \begin{pmatrix} 0 & 10 & 12 \\ 10 & 0 & 30 \\ 12 & 30 & 0 \end{pmatrix}.$$

These give $(q_1, q_2, q_3) = (14, 20, 15)$. Verify $q_1 = 14$ is correct, and then calculate an upper bound for this QKP. Briefly explain how you might implement a branch and bound algorithm to solve a QKP.

**3**

Suppose $n$ jobs are to be assigned to $n$ agents, one job per agent, to minimize cost, where the cost of assigning job $i$ to agent $j$ is $c_{ij}$. Formulate this problem as a minimum cost network flow problem.

Why might you not wish to use the network simplex algorithm to solve this problem?

Prove that a feasible solution $\{x_{ij}\}$ is optimal if there exist $\{\lambda_i\}$, $\{\mu_j\}$ such that $\lambda_i - \mu_j \leqslant c_{ij}$ for all $i, j$, and $\lambda_i - \mu_j = c_{ij}$ if $x_{ij} = 1$.

Explain how the Hungarian algorithm solves an assignment problem by adjusting feasible $\{\lambda_i\}$, $\{\mu_j\}$. Illustrate your explanation by solving the problem when

$$(c_{ij}) = \begin{pmatrix} 10 & 35 & 50 \\ 50 & 70 & 60 \\ 50 & 80 & 75 \end{pmatrix}.$$

**4**

Consider a bimatrix game in which both players have the same $m$ pure strategies. If the row and column players play pure strategies $i$ and $j$ then the payoffs to the row and column player are $p_{ij}$ and $p_{ji}$, respectively, where $P = (p_{ij})$ is a given $m \times m$ matrix. Explain what is meant by an equilibrium of this game.

Suppose $x, z \in \mathbb{R}^m$ can be found such that

$$x \geqslant 0,\ x \neq 0,\ z \geqslant 0,\ Px + z = \mathbf{1} \text{ and } x^T z = 0.$$

Prove that an equilibrium can be obtained from $x$.

Use the Lemke-Howson algorithm to find one equilibrium of the game with

$$P = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 2 \end{pmatrix}.$$

Are there any other equilibria?

**5**

State the Gibbard-Satterthwaite theorem. Define all terminology which you use in stating the theorem.

Suppose there are just 2 agents, $f$ is strategyproof and there exist a pair of linear orders $\succ_1, \succ_2$ such that alternative $a$ is ranked top by $\succ_1$, and ranked bottom by $\succ_2$, and $f$ selects $a$. Prove that $f$ must select $a$ for every $\succ_1', \succ_2'$ for which $\succ_1'$ ranks $a$ top.

Prove the Gibbard-Satterthwaite theorem in the case of two agents.

Explain why the theorem fails in the case of 3 agents and 2 alternatives.

**6**

A single item is to be auctioned amongst $n$ bidders who have symmetric independent private values that are uniformly distributed on $[0, 1]$. Assume the item is awarded to the highest bidder, that whatever his valuation a bidder will find it rational to bid, and in equilibrium the bids will be in the order of the valuations. Prove that the expected revenue that the seller will obtain from a bidder whose valuation is $\theta$ is $(n-1)\theta^n/n$.

Find the bid that will be made by a bidder having valuation $\theta$ if the auction is a sealed-bid auction in which the highest bidder wins, pays his bid, and losing bidders pay nothing.

Now suppose 2 items are to be auctioned amongst 3 bidders. The items are awarded in a sealed-bid auction to the two bidders who submit the two greatest bids. They pay their bids and the third bidder pays nothing. Show that a bidder whose valuation is $\theta$ will bid

$$\frac{3\theta - 2\theta^2}{6 - 3\theta}.$$

Find the expected revenue obtained by the seller and compare it to his expected revenue if he auctions only one item.

Consider the auctioning mechanism $(f, p)$ in which $f : (\theta_1, \theta_2, \theta_3) \to A = \{(1, 2), (1, 3), (2, 3)\}$ selects the two bidders who declare greatest two $\theta_i$, who are awarded items, and bidder $i$ pays

$$p_i(\theta) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) - \sum_{j \neq i} v_j(f(\theta), \theta_j)$$

where $v_j(a, \theta_j) = \theta_j$ or 0 as $j \in a$ or $j \notin a$, respectively.

Explain why this is a direct revelation mechanism. Show that the seller's expected revenue is the same as in the sealed-bid auction above.

## END OF PAPER