UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 27 May, 2016 1:30 pm to 4:30 pm

PAPER 211

ADVANCED FINANCIAL MODELS

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let W be a Brownian motion and let $S_t = S_0 e^{\mu t + \sigma W_t}$ for a real constant μ and positive constants σ, S_0.

(a) Find μ such that the process S is a martingale in its natural filtration.

For the rest of the question, let μ be such that S is a martingale. Further, define a function by

$$F(v,m) = \int (e^{-v/2+\sqrt{v}z} - m)^+ \phi(z) dz$$

for non-negative v, m where $\phi(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$ is the standard normal density.

(b) Fix positive constants T, K and let

$$C_t = S_t F\left((T-t)\sigma^2, \frac{K}{S_t}\right)$$

for $0 \leqslant t \leqslant T$. Show that C is a martingale.

Now let $\hat{S}_t = \mathbf{1}_{\{t \leqslant \tau\}} e^{\lambda t} S_t$ where τ is an exponential random variable with rate λ, independent of W.

(c) Show that $\hat{S}$ is a martingale in its natural filtration.

(d) Let $\hat{C}$ be a martingale in the filtration generated by $\hat{S}$, such that $\hat{C}_T = (\hat{S}_T - K)^+$. For any $0 \leqslant t \leqslant T$, express $\hat{C}_t$ in terms of the parameters λ, σ, T, K, the function F, and the random variable $\hat{S}_t$.

2

Consider a continuous-time risk-free bond market, and let $f(t,T)$ denote the forward rate at time t for maturity T.

(a) How is the spot interest rate r_t calculated in terms of the forward rates? How is the zero-coupon bond price $P(t,T)$ calculated in terms of the forward rates?

Suppose the forward rates evolve as

$$df(t,T) = \sigma(t,T) \int_t^T \sigma(t,u)du \; dt + \sigma(t,T)dW_t$$

where the function $(t,T) \to \sigma(t,T)$ is bounded, continuous and not random, and where W is a Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$.

(b) Show that the discounted bond price $e^{-\int_0^t r_s ds} P(t,T)$ is a martingale. You may use a version of the stochastic Fubini theorem without justification.

For each $T > 0$, define a measure $\mathbb{Q}_T$ on $(\Omega, \mathcal{F}_T)$ by

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}} = \frac{e^{-\int_0^T r_s ds}}{P(0,T)}.$$

(c) Show that the forward rate for maturity T is a $\mathbb{Q}_T$-martingale.

(d) Fix $0 < T_1 < T_2$. Express $\mathbb{E}^{\mathbb{Q}_{T_1}}[P(T_1,T_2)]$ in terms of the initial bond prices $P(0,T_1)$ and $P(0,T_2)$. Show that

$$\mathrm{Var}^{\mathbb{Q}_{T_1}}[\log P(T_1,T_2)] = \int_0^{T_1} \left(\int_{T_1}^{T_2} \sigma(t,u)du \right)^2 dt.$$

[You may use Itô's formula and Girsanov's theorem without proof.]

3

Let p be a given vector in $\mathbb{R}^n$, and let P be a bounded $\mathbb{R}^n$-valued random vector. Define a collection of random variables

$$\mathcal{Z} = \{Z : Z > 0 \text{ almost surely and } \mathbb{E}(ZP) = p\}$$

and suppose that $\mathcal{Z}$ is not empty.

(a) Suppose $H \in \mathbb{R}^n$ is not-random and such that $H \cdot p \leqslant 0 \leqslant H \cdot P$ almost surely. Prove that $H \cdot p = 0 = H \cdot P$.

Let X be a bounded random variable and x a constant such that

$$x \geqslant \mathbb{E}(ZX) \text{ for all } Z \in \mathcal{Z}.$$

For each $\gamma > 1$ and $H \in \mathbb{R}^n$ let

$$F_\gamma(H) = e^{\gamma(H \cdot p - x)} + \mathbb{E}[e^{\gamma(X - H \cdot P)}].$$

Assume for each γ, the function F_γ has a unique minimiser H_γ.

(b) Show that

$$\frac{\partial}{\partial \gamma} F_\gamma |_{H = H_\gamma} \leqslant 0$$

(c) Show there exists a non-random $H^* \in \mathbb{R}^n$ such that

$$x \geqslant H^* \cdot p \text{ and } H^* \cdot P \geqslant X \text{ almost surely.}$$

You may use without proof that $\sup_{\gamma > 1} F_\gamma(H_\gamma) < \infty$ and $\sup_{\gamma > 1} \|H_\gamma\| < \infty$.

Consider a two-asset, one-period market model, where the first asset is cash so that $B_0 = B_1 = 1$ and the second asset is a stock with $S_0 = 10$ and

$$\mathbb{P}(S_1 = 9) = \mathbb{P}(S_1 = 10) = \mathbb{P}(S_1 = 11) = \frac{1}{3}.$$

To this market, add a call option with strike $K = 10$ maturing at time 1.

(d) Find the super-replication strategy for the call with the smallest initial cost.

4

(a) What does it mean to say that a discrete-time market model is complete?

Consider a discrete-time model of a market with two assets: a numéraire with price process N and a stock with price process S. Suppose the market is complete, and that $N_{t+1} \geqslant N_t$ almost surely for all $t \geqslant 0$. Let $C(T, K)$ be the initial replication cost of a European call option on the stock with strike K and maturity T.

(b) Show that $T \mapsto C(T, K)$ is increasing for each $K > 0$.

(c) Compute $C(1, 18)$ in the case where $(N_0, S_0) = (10, 10)$ and

$$\mathbb{P}((N_1, S_1) = (15, 20)) = 1/2 = \mathbb{P}((N_1, S_1) = (20, 15))$$

Consider an option which matures at time T with payout $\left(\frac{1}{T}\sum_{t=1}^{T} S_t - K\right)^+$. [This is called an Asian option.] Let $A(T, K)$ be the initial replication cost.

(d) Show that $A(T, K) \leqslant \frac{1}{T}\sum_{t=1}^{T} C(t, K)$ for all $T > 0$ and $K > 0$.

5

Let $(Z_t)_{0 \leqslant t \leqslant T}$ be a given discrete-time integrable process adapted to the filtration $(\mathcal{F}_t)_{0 \leqslant t \leqslant T}$. Let $(U_t)_{0 \leqslant t \leqslant T}$ be its Snell envelope defined by

$$\begin{aligned} U_T &= Z_T \\ U_t &= \max\{Z_t, \mathbb{E}[U_{t+1}|\mathcal{F}_t]\} \text{ for } 0 \leqslant t \leqslant T-1. \end{aligned}$$

(a) Show that U is a supermartingale. Show that U is a martingale if Z is a submartingale.

Let $(S_t)_{0 \leqslant t \leqslant T}$ be such that the increments $S_1 - S_0, \ldots, S_T - S_{T-1}$ are independent and identically distributed, and let the filtration be generated by S. Fix a measurable function $f : \mathbb{R} \to \mathbb{R}$ and let $Z_t = f(S_t)$. Suppose that Z_t is integrable for each $t \geqslant 0$, and let U be the Snell envelope of Z.

(b) Show that there exists a deterministic function V such that $U_t = V(t, S_t)$.

(c) Prove that if the function f is convex then the functions $V(t, \cdot)$ are convex for each $0 \leqslant t \leqslant T$.

[Recall that a function $\varphi : \mathbb{R} \to \mathbb{R}$ is called convex if

$$\varphi[\theta x + (1-\theta)y] \leqslant \theta\varphi(x) + (1-\theta)\varphi(y)$$

for all $x, y \in \mathbb{R}$ and $0 < \theta < 1$.]

6

Suppose $(W_t)_{t \geqslant 0}$ is a Brownian motion and $(S_t)_{t \geqslant 0}$ evolves as

$$dS_t = a(S_t) dW_t.$$

Let $V : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ be the unique solution to

$$\frac{\partial}{\partial t} V(t,S) + \frac{a(S)^2}{2} \frac{\partial^2}{\partial S^2} V(t,S) = 0$$
$$V(T,S) = g(S) \text{ for all } S \in \mathbb{R}.$$

Finally, let $\xi_t = V(t, S_t)$ for $0 \leqslant t \leqslant T$. Assume that the functions a, V, and g are smooth and bounded with bounded derivatives.

(a) Show that

$$\xi_t = \mathbb{E}[g(S_T)|\mathcal{F}_t]$$

where $(\mathcal{F}_t)_{t \geqslant 0}$ is the filtration generated by the Brownian motion.

Let $U : [0,T] \times \mathbb{R} \to \mathbb{R}$ be the unique solution to

$$\frac{\partial}{\partial t} U(t,S) + a(S)a'(S) \frac{\partial}{\partial S} U(t,S) + \frac{a(S)^2}{2} \frac{\partial^2}{\partial S^2} U(t,S) = 0$$
$$U(T,S) = g'(S) \text{ for all } S \in \mathbb{R}.$$

Let $\pi_t = U(t, S_t)$ for $0 \leqslant t \leqslant T$. Assume U is smooth and bounded with bounded derivatives.

(b) Show that

$$\xi_t = V(0, S_0) + \int_0^t \pi_s dS_s.$$

Let $(Z_t)_{t \geqslant 0}$ be the martingale defined by $Z_0 = 1$ and

$$dZ_t = Z_t a'(S_t) dW_t.$$

and define an equivalent measure $\hat{\mathbb{P}}$ with density Z_T.

(c) Show that

$$\pi_t = \mathbb{E}^{\hat{\mathbb{P}}}(g'(S_T)|\mathcal{F}_t).$$

(d) Briefly comment on the financial significance of the random variables ξ_t and π_t in the context of a market with stock price $(S_t)_{t \geqslant 0}$.

[You may use Itô's formula and Girsanov's theorem without proof.]

END OF PAPER