UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2016 1:30 pm to 3:30 pm

PAPER 209

STATISTICS FOR STOCHASTIC PROCESSES

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the process $X_t = x_0 + \int_0^t \sigma(s)\mathrm{d}W_s$, $t \in [0,1]$, $x_0 \in \mathbb{R}$, where $(W_s)_{s \geqslant 0}$ is Brownian motion and $\sigma : [0,1] \to \mathbb{R}$ is a measurable, bounded, deterministic function. For integers $n \geqslant 1$ the process $(X_t)_{t\in[0,1]}$ is observed at discrete times $0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = 1$. For $i \in \{1,\ldots,n\}$ define $\Delta X_{i,n} := X_{t_{i,n}} - X_{t_{i-1,n}}$, $\Delta t_{i,n} := t_{i,n} - t_{i-1,n}$ and $\Delta_n := \max_{1\leqslant i\leqslant n} \Delta t_{i,n}$. Let $g : [0,1] \to \mathbb{R}$ be a measurable function bounded by $R > 0$.

(a) Let $M_n := \sum_{i=1}^n g(t_{i-1,n})[(\Delta X_{i,n})^2 - \int_{t_{i-1,n}}^{t_{i,n}} \sigma^2(s)\mathrm{d}s]$. Show that there exists an absolute constant $D > 0$ such that

$$\mathbb{E}[M_n^2] \leqslant DR^2\|\sigma^4\|_\infty \Delta_n.$$

Now consider the estimator $\hat{\Lambda}_n(g) := \sum_{i=1}^n g(t_{i-1,n})(\Delta X_{i,n})^2$ for $\Lambda(g) := \int_0^1 g(s)\sigma^2(s)\mathrm{d}s$.

(b) Let g satisfy in addition for some $\alpha \in (0,1]$ that $|g(t) - g(s)| \leqslant R|t-s|^\alpha$. Show that there exists an absolute constant $\tilde{D} > 0$ such that

$$\mathbb{E}[(\hat{\Lambda}_n(g) - \Lambda(g))^2] \leqslant \tilde{D}R^2\|\sigma^4\|_\infty \max(\Delta_n, \Delta_n^{2\alpha}).$$

(c) Suppose that there are constants $C > 0$ and $\beta \in (0,1]$ such that $|\sigma^2(s) - \sigma^2(t)| \leqslant C|t-s|^\beta$ for all $s,t \in [0,1]$. Define $g_n := \frac{1}{h_n}\mathbb{1}_{[t_0,t_0+h_n]}$ for $0 < t_0 < t_0 + h_n < 1$ and $h_n = \Delta_n^{1/(2+2\beta)}$. Show that there are constants $D_1, D_2 > 0$ depending only on C such that

$$\mathbb{E}[(\hat{\Lambda}_n(g_n) - \Lambda(g_n))^2] \leqslant D_1\|\sigma^4\|_\infty \frac{\Delta_n}{h_n^2},$$

$$\mathbb{E}[(\hat{\Lambda}_n(g_n) - \sigma(t_0)^2)^2] \leqslant (D_1\|\sigma^4\|_\infty + D_2)\Delta_n^{2\beta/(2\beta+2)}.$$

[Hint: You may use without proof that $\Delta X_{i,n} \sim N(0, \int_{t_{i-1,n}}^{t_{i,n}} \sigma^2(s)\mathrm{d}s)$ and that $\Delta X_{i,n}$, $i \in \{1,\ldots,n\}$, are independent.]

2

(a) Define the *empirical characteristic function* φ_n and the *empirical characteristic process* $\mathcal{C}_n$ for i.i.d. real-valued random variables $X_1, \ldots, X_n$.

(b) Let $X_1, \ldots, X_n$ be i.i.d. real-valued random variables. Let φ be the characteristic function of X_1 and φ_n be the empirical characteristic function of $X_1, \ldots, X_n$. If $\mathrm{Cov}_{\mathbb{C}}(Z_1, Z_2) := \mathbb{E}[Z_1\overline{Z_2}] - \mathbb{E}[Z_1]\overline{\mathbb{E}[Z_2]}$ and $\mathrm{Var}_{\mathbb{C}}(Z_1) := \mathbb{E}[|Z_1 - \mathbb{E}[Z_1]|^2]$ for complex-valued random variables Z_1 and Z_2, show that $\mathrm{Cov}_{\mathbb{C}}(\varphi_n(u), \varphi_n(v)) = \frac{1}{n}(\varphi(u-v) - \varphi(u)\varphi(-v))$ and $\mathrm{Var}_{\mathbb{C}}(\varphi_n(u)) \leqslant \frac{1}{n}$ for all $u, v \in \mathbb{R}$.

(c) For $u \in \mathbb{R}$ let $\Gamma(u)$ be a complex-valued random variable such that $(\mathrm{Re}(\Gamma(u)), \mathrm{Im}(\Gamma(u)))$ is a centred normal random variable in $\mathbb{R}^2$ such that $\mathrm{Cov}_{\mathbb{C}}(\Gamma(u), \Gamma(u)) = 1 - |\varphi(u)|^2$ and $\mathrm{Cov}_{\mathbb{C}}(\Gamma(u), \overline{\Gamma(u)}) = \varphi(2u) - \varphi(u)^2$. Show that for all $u \in \mathbb{R}$ the empirical characteristic process $\mathcal{C}_n$ satisfies $(\mathrm{Re}(\mathcal{C}_n(u)), \mathrm{Im}(\mathcal{C}_n(u))) \to (\mathrm{Re}(\Gamma(u)), \mathrm{Im}(\Gamma(u)))$ as $n \to \infty$ in distribution.

[*Hint: You may use without proof that if* $(Y_n)_{n\in\mathbb{N}}$ *are centred i.i.d. random vectors in* $\mathbb{R}^d$, *then* $\frac{1}{\sqrt{n}}\sum_{k=1}^n Y_k$ *converges as* $n \to \infty$ *in distribution to a centred normal random vector with covariance* $\mathbb{E}[Y_1 Y_1^\top]$.]

(d) Let X and ϵ be independent real-valued random variables whose distributions are absolutely continuous with respect to the Lebesgue measure with Lebesgue densities p_X and p_ϵ, respectively. Let φ_ϵ and φ^Y be the characteristic functions of ϵ and $Y = X + \epsilon$, respectively. Let φ_n^Y be the empirical characteristic function of n i.i.d. copies of Y. Suppose $\varphi_\epsilon(u) \neq 0$ for all $u \in \mathbb{R}$ and let $\tilde{M}_h := \sup_{|u|\leqslant 1/h} |1/\varphi_\epsilon(u)|$.

Let $K : \mathbb{R} \to \mathbb{R}$ be an integrable function such that $\int_{-\infty}^{\infty} K(x)\mathrm{d}x = 1$. Define $K_h(x) := \frac{1}{h}K(\frac{x}{h})$ for $h > 0$. Let $\mathcal{F}$ denote the Fourier transform, i.e., $\mathcal{F}f(u) = \int_{-\infty}^{\infty} f(x)e^{iux}\mathrm{d}x$ for all integrable functions f. Further assume $\mathrm{supp}(\mathcal{F}K) \subseteq [-1, 1]$. Show that

$$\left(\int_{-\infty}^{\infty} \left|\frac{\varphi_n^Y(u) - \varphi^Y(u)}{\varphi_\epsilon(u)}\right|^2 |\mathcal{F}[K_h](u)|^2 \mathrm{d}u\right)^{1/2} = O_{\mathbb{P}}\left(\frac{\tilde{M}_h}{\sqrt{hn}}\right).$$

3

Let the function $b : \mathbb{R} \to \mathbb{R}$ be such that for some $\alpha \in (0,1]$ and $R > 0$, we have $|b(x) - b(y)| \leqslant R|x-y|^\alpha$. Let further $\sigma : \mathbb{R} \to \mathbb{R}$ be a measurable and bounded function with $\inf_{x\in\mathbb{R}} \sigma^2(x) \geqslant \underline{\sigma}^2$ for some $\underline{\sigma}^2 > 0$. Suppose that there are $M, \gamma > 0$ such that $2b(x)/\sigma^2(x) \geqslant \gamma$ for all $x \leqslant -M$ and $2b(x)/\sigma^2(x) \leqslant -\gamma$ for all $x \geqslant M$. Let $(X_t)_{t\geqslant 0}$ be a stationary, strong solution of the stochastic differential equation $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$, $t \geqslant 0$, where $(W_t)_{t\geqslant 0}$ is a Brownian motion. Define for $h > 0$ and $T \geqslant 0$

$$\hat{b}_T(x,h) = \frac{\int_0^T \mathbb{1}_{[x-h,x+h]}(X_t)\mathrm{d}X_t}{\int_0^T \mathbb{1}_{[x-h,x+h]}(X_t)\mathrm{d}t} \qquad \text{if } \int_0^T \mathbb{1}_{[x-h,x+h]}(X_t)\mathrm{d}t > 0$$

and $\hat{b}_T(x,h) = 0$ otherwise. Show that there exist $T_0, h_0 > 0$ such that for all $h \in (0, h_0)$ and for all $T \geqslant T_0$

$$\sup_{x\in[-M/2,M/2]} |\hat{b}_T(x,h) - b(x)| \leqslant Rh^\alpha + O_{\mathbb{P}}\left(\frac{1}{\sqrt{Th}}\right).$$

[Hint: You may use without proof the Itô isometry and that the invariant measure μ is unique, its density $\tilde{\mu}$ is bounded and $1/\tilde{\mu}$ is locally bounded. Further you may use without proof that there exists a constant $C > 0$ such that for all $T \geqslant 0$ and for all μ-integrable functions f with $\int_{-\infty}^{\infty} f(x)\mathrm{d}\mu(x) = 0$

$$\mathbb{E}\left[\left(\int_0^T f(X_t)\mathrm{d}t\right)^2\right] \leqslant C(1+T)\left(\left(\int_{-\infty}^{\infty} |f(x)|\mathrm{d}\mu(x)\right)^2 + \sup_{|x|\geqslant M} |f(x)|^2\right). \quad]$$

4

Let $(X_k)_{k\geqslant 1}$ be i.i.d. random vectors in $\mathbb{R}^d$ such that $\mathbb{E}[\|X_1\|] < \infty$, where $\|X_1\|$ denotes the Euclidean norm of X_1. Set $S_n(u) = \sum_{k=1}^n (\cos(\langle X_k, u\rangle) - \mathbb{E}[\cos(\langle X_k, u\rangle)])$ for $u \in \mathbb{R}^d$. Show that there exists some constant C depending on d and $\mathbb{E}[\|X_1\|]$ only such that S_n satisfies for all $K \geqslant 2$, for all integers $n \geqslant 1$ and for all $R > 8\sqrt{d}$

$$\mathbb{P}\left(\max_{u\in[-K,K]^d} |S_n(u)| \geqslant \frac{R}{2}\sqrt{n\log(nK^2)}\right) \leqslant C(\sqrt{n}K)^{(64d-R^2)/(64d+64)}.$$

[Hint: You may use without proof that if $Y_1, \ldots, Y_n$ are real-valued, centred, i.i.d. random variables and $|Y_1| \leqslant M$ almost surely for some $M > 0$, then for all $\tau > 0$

$$\mathbb{P}\left(\left|\sum_{k=1}^n Y_k\right| \geqslant \tau\right) \leqslant 2\exp\left(-\frac{\tau^2}{2nM^2}\right). \quad]$$

END OF PAPER