UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2016 9:00 am to 11:00 am

PAPER 204

PERCOLATION AND RELATED TOPICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Explain what is meant by the *bond percolation model* on an undirected graph $G = (V, E)$.

(b) Define an *increasing event*. State the FKG inequality in this setting, and also the BK disjoint-occurrence inequality for a pair of increasing events, being careful to define any terms used.

(c) Let G be the square lattice, and let $\Lambda_n = [-n, n]^2$ and $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$. Show that $g_k := P_p(0 \leftrightarrow \partial\Lambda_k)$ satisfies

$$g_{m+n} \leqslant \left|\partial\Lambda_m\right| g_m g_n, \qquad m, n \geqslant 0.$$

Deduce that the limit $\gamma = \lim_{n\to\infty} g_n^{1/n}$ exists and satisfies $p \leqslant \gamma \leqslant 1$. (Any standard results which you use should be stated clearly.)

(d) Let $h_k = P_p(0 \leftrightarrow e_k)$ where $e_k = (k, 0)$. Show that

$$\left(\frac{g_k}{8k}\right)^2 \leqslant h_{2k} \leqslant g_{2k}, \qquad k \geqslant 1,$$

and deduce that $\lim_{n\to\infty} h_n^{1/n} = \gamma$. (The cases of even/odd n may be treated separately.)

2

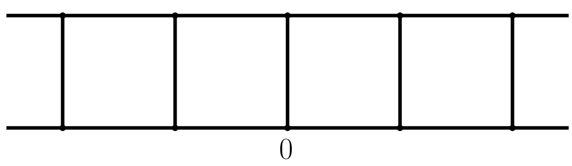
(a) Define an *n-step self-avoiding walk* (SAW) on a graph G.

(b) Let σ_n be the number of n-step SAWs on the doubly-infinite ladder graph (illustrated below) that start at 0 and at each step move either vertically or to the right. Show that $\sigma_n^{1/n} \to \gamma$, where $\gamma = \frac{1}{2}(1 + \sqrt{5})$ is the golden mean.

(c) Consider bond percolation on the square lattice $\mathbb{L}^2$ with parameter p. Show that the critical probability p_c satisfies $p_c \leqslant 1 - \mu^{-1}$, where μ is the connective constant of $\mathbb{L}^2$.

(d) Let κ_n be the (random) number of n-step SAWs of $\mathbb{L}^2$ starting at 0 that use open edges only. Show that $\limsup_{n\to\infty} E_p(\kappa_n^{1/n}) \leqslant p\mu$, where E_p denotes expectation.

(*You may use Lyapunov's inequality, namely,* $\{E(|Z^r|\}^{1/r}$ *is non-decreasing in* $r = 0, 1, 2 \ldots$.)



3

(a) Define the *percolation probability* $\theta(p)$ of the site percolation model on the d-dimensional cubic lattice $\mathbb{L}^d = (V, E)$ with parameter p.

(b) Show that the function θ is *right continuous* in that $\lim_{p' \downarrow p} \theta(p') = \theta(p)$.

(c) Explain how to construct a sequence of processes $X_p = (X_p(v) : v \in V)$, for $p \in [0,1]$, satisfying: (i) $X_r \leqslant X_s$ for $r \leqslant s$, and (ii) for $p \in [0,1]$, X_p is a site model with parameter p, as in part (a).

(d) Let I_p be the event that X_p contains an infinite open path starting at the origin. Show that

$$\lim_{p' \uparrow p} \theta(p') = \theta(p) - \mathbb{P}\big(I_p \cap \{M = p\}\big),$$

where $M = \inf\{p : I_p \text{ occurs}\}$ and $\mathbb{P}$ is the appropriate probability measure.

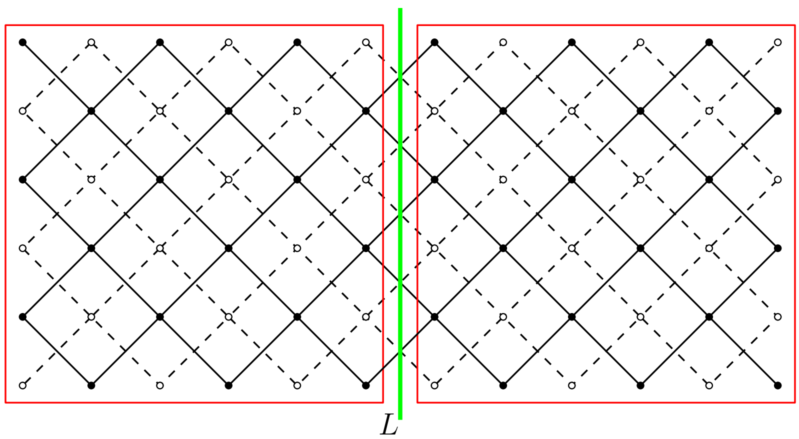
(e) Deduce that the function θ is *continuous* on the half-open interval $(p_{\mathrm{c}}, 1]$. (Any general result may be used without proof, but should be stated carefully.)

4

Consider bond percolation on the square lattice $\mathbb{L}^2$ with parameter p, and rotate $\mathbb{L}^2$ through the angle $\pi/4$ as illustrated in the figure below (which contains both $\mathbb{L}^2$ and its dual). Let $\theta(p)$ denote the percolation probability.

(a) Let $R_{m,n}$ be a rectangle of $\mathbb{L}^2$ containing m vertices on each horizontal side, and n on each vertical side (two rectangles of shape $R_{3,3}$ form a rectangle of shape $R_{6,3}$, as indicated in the figure). Let $C_{m,n}$ be the event that $R_{m,n}$ is crossed from left to right by an open path. By considering reflection in the line L or otherwise, show that there exists $\tau > 0$ such that $P_{\frac{1}{2}}(C_{3m,m}) > \tau$ for all $m \geqslant 1$.

(b) Deduce that there exist constants $A < \infty$ and $\beta > 0$ such that $\theta(p) \leqslant A(p - \frac{1}{2})^\beta$ for $p > \frac{1}{2}$.



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