UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 6 June, 2016 9:00 am to 11:00 am

PAPER 203

SCHRAMM-LOEWNER EVOLUTIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let $\phi : D \to D'$ be a conformal isomorphism of planar domains and assume that D is bounded. Fix $z \in D$ and set $z' = \phi(z)$. Let $(B_t)_{t \geqslant 0}$ and $(B'_t)_{t \geqslant 0}$ be complex Brownian motions starting from z and z' respectively. Set

$$T = \inf\{t \geqslant 0 : B_t \notin D\}, \quad T' = \inf\{t \geqslant 0 : B'_t \notin D'\}.$$

Show that, for a suitable random homeomorphism $\tau : [0, T) \to [0, \tilde{T})$ and for $\tilde{B}_t = \phi(B_{\tau(t)})$, the random processes $(\tilde{T}, (\tilde{B}_t)_{t<\tilde{T}})$ and $(T', (B'_t)_{t<T'})$ have the same distribution.

(b) Deduce that the random time T' is almost surely finite.

(c) Does there exist a conformal isomorphism $\phi : D \to \mathbb{C} \setminus \{0\}$? Justify your answer.

(d) Define the Green function G_D for D and show that, for all $w \in D$ with $w \neq z$ and for $w' = \phi(w)$, we have

$$G_{D'}(z', w') = G_D(z, w).$$

2

Let K be a compact $\mathbb{H}$-hull and set $H = \mathbb{H} \setminus K$. Assume that K is contained in the closed unit ball $\bar{\mathbb{D}}$.

(a) What does it mean to say that g_K is the mapping-out function for K?

(b) Let $y \in (1, \infty)$ and let B be a complex Brownian motion starting from iy. Set

$$T = \inf\{t \geqslant 0 : B_t \notin H\}.$$

Let $x, b \in (1, \infty)$ with $x < b$. Show that, in the limit $y \to \infty$, we have

$$\pi y \mathbb{P}_{iy}(B_T \in (x, b)) \to g_K(b) - g_K(x).$$

You may assume standard properties of the mapping-out function provided these are clearly stated.

(c) Show that $g_K(x) \in [x, x + 1/x]$ for all $x \in (1, \infty)$.

(d) Show that $|g_K(z) - z| \leqslant 3$ for all $z \in H$.

(e) Give an example to show that the constant 3 cannot be improved. A detailed justification of this is not required.

3

(a) Let $\gamma = (\gamma_t)_{t \geqslant 0}$ be a continuous random process in the closed upper half-plane $\bar{\mathbb{H}}$ with $\gamma_0 = 0$. What does it mean to say that γ is a Schramm–Loewner evolution of parameter $\kappa \in [0, \infty)$?

(b) Let γ be an SLE(κ) and let $b \in (0, \infty)$. Show how the event that γ hits the interval $[b, \infty) \subseteq \mathbb{R}$ may be characterized in terms of the Loewner flow on $\mathbb{R}$.

(c) Show that, for some $\kappa_c \in (0, \infty)$, to be determined, we have

$$\mathbb{P}(\gamma \text{ hits } [b, \infty)) = \begin{cases} 0, & \text{for } \kappa < \kappa_c, \\ 1, & \text{for } \kappa > \kappa_c. \end{cases}$$

(d) Hence show that, if $\kappa < \kappa_c$ then almost surely γ hits no point in $(0, \infty)$, while if $\kappa > \kappa_c$ then almost surely γ hits an unbounded set of points in $(0, \infty)$.

(e) What happens when $\kappa = \kappa_c$?

You may use without proof any general fact from the deterministic Loewner theory, provided that you state it clearly.

4

(a) Let ϕ be a conformal automorphism of $\mathbb{H}$ such that $\phi(0) = 0$ and which extends to a homeomorphism of initial domains $\mathbb{H} \cup (-1, \infty) \to \mathbb{H} \cup (\infty, 1)$. Find an explicit formula for ϕ.

(b) Let γ be an SLE(6) in $(\mathbb{H}, 0, \infty)$ and set $\tilde{\gamma}_t = \phi(\gamma_t)$. Show that $\tilde{\gamma}$ is an SLE(6) in $(\mathbb{H}, 0, 1)$ of scale $\sigma(w) = w/(1-w)$.

(c) Write K_t and $\tilde{K}_t$ for the compact $\mathbb{H}$-hulls generated by $\gamma(0, t]$ and $\tilde{\gamma}(0, t]$ respectively, and write g_t and $\tilde{g}_t$ for their mapping out functions. Set

$$T = \inf\{t \geqslant 0 : \gamma_t \in (-\infty, -1]\}$$

and, for $t < T$, consider the conformal automorphism of $\mathbb{H}$ given by

$$\phi_t = \tilde{g}_t \circ \phi \circ g_t^{-1}.$$

You may assume that the family of compact $\mathbb{H}$-hulls $(\tilde{K}_t)_{t<T}$ has the local growth property and has Loewner transform $(\tilde{\xi}_t)_{t<T}$ given by $\tilde{\xi}_t = \phi_t(\xi_t)$. You may also assume that

$$\mathrm{hcap}(\tilde{K}_t) = 2 \int_0^t \phi_s'(\xi_s)^2 ds.$$

Compute the time derivative $\dot{\phi}_t(z)$ for $z \in \mathbb{H}$ and hence deduce that

$$\dot{\phi}_t(\xi_t) = -3\phi_t''(\xi_t).$$

(d) Show that $\tilde{\gamma}$ is also a time-change of an SLE(6) in $(\mathbb{H}, 0, \infty)$ up to time T.

END OF PAPER