UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 1 June, 2016 1:30 pm to 4:30 pm

PAPER 202

STOCHASTIC CALCULUS

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Give the definition of the *previsible σ-algebra* $\mathcal{P}$, a *previsible process*, and a *simple process.*

(b) Show that if H is a previsible process then H_t is $\mathcal{F}_{t-}$-measurable for all $t \geqslant 0$.

(c) Let μ be a finite measure on $\mathcal{P}$. Show that the set $\mathcal{S}$ of simple processes is dense in $L^2(\mathcal{P}, \mu)$.

(d) Suppose that H is a simple process and that M is a bounded, continuous martingale. Give the definition of the *stochastic integral* $H \cdot M$. Express $\mathbb{E}[(H \cdot M)_t^2]$ in terms of the values of H and the squares of the increments of M. State the Itô isometry and explain how the formula you have obtained in the previous part is related to the Itô isometry.

2

(a) Give the definition of what it means for a probability measure $\widetilde{\mathbb{P}}$ to be *absolutely continuous* with respect to another probability measure $\mathbb{P}$.

(b) Give the definition of the *stochastic exponential* of a continuous local martingale M and state the Girsanov theorem. Can the quadratic variation of a continuous local martingale be changed when performing a change of measures as in the Girsanov theorem?

(c) Suppose that M is a continuous local martingale with $M_0 = 0$. Show that the existence of a constant $C > 0$ such that $[M]_\infty \leqslant C$ is a sufficient condition to apply the Girsanov theorem.

(d) Suppose that $dX_t = (\delta - 1)/2 \cdot X_t^{-1} dt + dB_t$ where B is a standard Brownian motion and $\delta \in \mathbb{R}$ is a constant. Assume that $X_0 = x$ with $0 < a < x < b < \infty$. For each $r > 0$, let $\tau_r = \inf\{t \geqslant 0 : X_t = r\}$. Fix $t > 0$. Write down the Radon-Nikodym derivative of the law of $X|_{[0,\tau_a \wedge \tau_b \wedge t]}$ with respect to the law of a standard Brownian motion and explain why the Girsanov theorem is applicable in this setting.

3

(a) Give the definition of a *solution* to the SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt.$$

Give the definition of a *strong solution*, *uniqueness in law*, and *pathwise uniqueness.*

(b) Suppose that B is a standard Brownian motion and b is a bounded, measurable function. Using the Girsanov theorem, construct a solution to the SDE:

$$dX_t = b(X_t)dt + dB_t.$$

Prove that uniqueness in law holds for this SDE.

(c) Without giving a proof, explain why X_t in the previous part may not be a strong solution.

4

(a) Give the definition of a *continuous martingale* and a *continuous local martingale.* Explain what it means for a filtered probability space to satisfy the *usual conditions.*

(b) State (without proof) Itô's formula and the integration by parts formula for semi-martingales

(c) Suppose that $M, N \in \mathcal{M}_{c,loc}$. Show that the *covariation process* $[M, N]$ exists and is unique. That is, show that there exists a unique, continuous, adapted process $[M, N]$ of finite variation such that $MN - [M, N] \in \mathcal{M}_{c,loc}$. State any results used about continuous local martingales and quadratic variation.

(d) State and prove the Lévy characterization of Brownian motion

5

(a) Suppose that D, D' are planar domains and that $f\colon D \to D'$ is a conformal map. Suppose that B is a standard Brownian motion in D starting from $z \in D$. Show that $f(B)$ is a time-change of a standard Brownian motion in D' starting from $f(z)$, defined up until the first time that it exits D'.

(b) Using that the first exit distribution of Brownian motion on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ starting from $z = x + iy$ has density with respect to Lebesgue measure on $\mathbb{R}$ at u given by

$$\frac{1}{\pi} \cdot \frac{y}{(x-u)^2 + y^2},$$

give the density $\rho(a)$ of the first exit distribution of Brownian motion on the strip $\{z \in \mathbb{C} : \text{Im}(z) \in [0, \pi]\}$ for a on the bottom part of the strip (i.e., $a \in \mathbb{R}$). [Hint: the map $z \mapsto e^z$ is a conformal map from the strip to $\mathbb{H}$.]

(c) Suppose that u is a harmonic function on $\mathbb{C}$ and that B is a standard Brownian motion in $\mathbb{C}$. Show that $u(B_t)$ is a continuous local martingale. Use this to show that if u is a bounded harmonic function on $\mathbb{C}$ then u is constant.

6

(a) Give the definition of an L-*diffusion*.

(b) Suppose that X is an L-diffusion and that $f \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$. Show that the process

$$M_t^f = f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + L \right) f(s, X_s) ds$$

is a martingale.

(c) Recall that the SDE for a Bessel process of dimension δ is

$$dX_t = \frac{\delta - 1}{2} \cdot \frac{1}{X_t} dt + dB_t.$$

Suppose that B is a standard Brownian motion and fix $a \in \mathbb{R}$. Compute the Itô derivative of $Z_t = e^{B_t + at}$ and derive an SDE for Z. For each $s \geqslant 0$, let $\tau_s = \inf\{t \geqslant 0 : [Z]_t > s\}$. Show that the process $s \mapsto Z_{\tau_s}$ is a Bessel process of dimension $2 + 2a$, up until the first time that it hits 0.

(d) Suppose that Z_t is a Bessel process of dimension δ, fix $r > 0$, and let $W_t = Z_t^r = (Z_t)^r$. For each $s \geqslant 0$, let $\sigma_s = \inf\{t \geqslant 0 : [W]_t > s\}$. Show that $s \mapsto W_{\sigma_s}$ is a Bessel process of dimension $2 + (\delta - 2)/r$, up until the first time that it hits 0.

END OF PAPER