UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 1 June, 2016 9:00 am to 12:00 pm

PAPER 201

ADVANCED PROBABILITY

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) What does it mean to say that a random process $(Y_n)_{n\geqslant 0}$ is a submartingale?

(b) State the almost sure martingale convergence theorem.

(c) Give an example of a non-negative martingale $(M_n)_{n\geqslant 0}$ such that $M_0 = 1$ and $M_n \to 0$ as $n \to \infty$.

(d) Let G be a connected graph with vertex set V and all vertex degrees finite. The random walk on G is the Markov chain $(X_n)_{n\geqslant 0}$ on V which moves at each step to an adjacent vertex, chosen uniformly at random. We say that the graph is recurrent if the random walk visits every vertex infinitely often almost surely, that is,

$$\mathbb{P}(X_n = v \text{ infinitely often for all } v) = 1.$$

Recall that a function $h : V \to \mathbb{R}$ is harmonic if

$$h(v) = \frac{1}{\deg(v)} \sum_{u\sim v} h(u)$$

where the sum is over all vertices adjacent to v. Prove that every bounded harmonic function on the vertices of a recurrent graph is constant.

(e) Let $(M_n)_{n\geqslant 0}$ be a martingale with bounded increments. Show that

$$\lim_n \frac{M_n}{n} = 0 \quad \text{almost surely.}$$

Show further that, if $(Y_n)_{n\geqslant 0}$ is a submartingale with bounded increments, then

$$\liminf_n \frac{Y_n}{n} \geqslant 0 \quad \text{almost surely.}$$

You may use the strong law of large numbers for uncorrelated random variables which states the following. Let $(X_n)_{n\geqslant 0}$ be a sequence of random variables, bounded in $L^2(\mathbb{P})$ and such that $\mathbb{E}(X_m X_n) = \mathbb{E}(X_m)\mathbb{E}(X_n)$ all distinct $m, n \geqslant 0$. Then

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - E(X_k)) \to 0 \quad \textit{almost surely.}$$

2

Let $(M_n)_{n\geqslant 0}$ be a non-negative martingale with filtration $(\mathcal{F}_n)_{n\geqslant 0}$. Set

$$T = \min\{n \geqslant 0 : M_n = 0\}.$$

(a) Show that $M_n = 0$ for all $n \geqslant T$ almost surely.

(b) Suppose now that $M_0 = 1$ and consider for $R > 0$ the stopping time

$$T_R = \min\{n \geqslant 0 : M_n \geqslant R\}.$$

Show that $\mathbb{P}(T_R < \infty) \leqslant 1/R$.

(c) Suppose further that, for some constants $\sigma \in (0,\infty)$ and $\tau \in [1,\infty)$,

$$\mathbb{E}((M_{n+1} - M_n)^2|\mathcal{F}_n) \geqslant \sigma^2 1_{\{T>n\}} \quad \text{and} \quad M_{T_R} \leqslant \tau R$$

for all n and R almost surely. Set $Y_n = M_n^2 - \sigma^2 n$ and $X_n = Y_{T\wedge T_R \wedge n}$. Show that $(X_n)_{n\geqslant 0}$ is a submartingale and deduce that

$$\mathbb{E}(T \wedge T_R) \leqslant \frac{\tau^2 R}{\sigma^2}.$$

(d) Hence show that, for all $n \geqslant 1$,

$$\mathbb{P}(T > n) \leqslant \frac{2\tau}{\sigma\sqrt{n}}.$$

3

(a) State Cramér's theorem on the probabilities $\mathbb{P}(S_n \geqslant na)$ associated to the partial sums $S_n = X_1 + \cdots + X_n$ of a sequence of independent, identically distributed, integrable real-valued random variables $(X_n : n \in \mathbb{N})$.

(b) Prove the upper bound in Cramér's theorem.

(c) Show that, for all $a \geqslant \mathbb{E}(X_1)$ such that $\mathbb{P}(X_1 > a) > 0$, and for all $\varepsilon > 0$,

$$\mathbb{P}(S_n/n \in [a, a+\varepsilon)|S_n/n \geqslant a) \to 1 \quad \text{as} \quad n \to \infty.$$

You may assume that the cumulant generating function ψ of X_1 is everywhere finite and may use any standard properties of ψ or of related functions.

4

(a) What does it mean to say that $(B_t)_{t \geqslant 0}$ is a Brownian motion?

(b) Define for each integer $n \geqslant 0$

$$[B]_n = \sum_{k=0}^{2^n-1} (B_{(k+1)2^{-n}} - B_{k2^{-n}})^2.$$

Show that $[B]_n$ converges in $L^2(\mathbb{P})$ as $n \to \infty$ and determine its limit. You may use without proof the fact that $\operatorname{var}(B_1^2) = 2$.

(c) Define for $w \in C[0,1]$ and $\alpha \in (0,1)$

$$\|w\|_\alpha = \sup_{s,t \in [0,1],\, s<t} \frac{|w(t) - w(s)|}{(t-s)^\alpha}.$$

Set

$$C^\alpha[0,1] = \{w \in C[0,1] : \|w\|_\alpha < \infty\}$$

and equip $C^\alpha[0,1]$ with the σ-algebra generated by the coordinate functions $X_t(w) = w(t)$. Show that for $\alpha > 1/2$ there does not exist a probability measure on $C^\alpha[0,1]$ such that $(X_t)_{0 \leqslant t \leqslant 1}$ is a Brownian motion.

(d) Show, on the other hand, that for $\alpha < 1/2$ there does exist a probability measure on $C^\alpha[0,1]$ such that $(X_t)_{0 \leqslant t \leqslant 1}$ is a Brownian motion. You may assume the existence of a probability measure on $C[0,1]$ for which $(X_t)_{0 \leqslant t \leqslant 1}$ is a Brownian motion.

5

Throughout this question, you may assume the optional stopping theorem.

(a) Show that a real Brownian motion exits every interval almost surely.

(b) Show that a two-dimensional Brownian motion hits every non-empty open ball almost surely.

(c) Consider the domain D in $\mathbb{R}^2$ given by

$$D = \{(x, y) \in \mathbb{R}^2 : \min\{|x|, |y|\} < 1\}$$

and define a function f on the boundary ∂D by

$$f(x, y) = \begin{cases} a, & \text{if } \min\{x, y\} = 1, \\ b, & \text{if } \min\{-x, y\} = 1, \\ c, & \text{if } \min\{-x, -y\} = 1, \\ d, & \text{if } \min\{x, -y\} = 1. \end{cases}$$

Let $(B_t)_{t \geqslant 0}$ be a Brownian motion in $\mathbb{R}^2$ starting from $z \in D$ and set

$$T = \inf\{t \geqslant 0 : B_t \in \partial D\}.$$

Suppose that u is a bounded harmonic function in D which is continuous on the closure $\bar{D}$ with boundary values given by f. Show that

$$u(z) = \mathbb{E}_z(f(B_T)).$$

(d) Show further that, for all $x \in (-1, 1)$, as $y \to \infty$,

$$u(x, y) \to \frac{a+b}{2} + \frac{a-b}{2} x.$$

6

(a) What does it mean to say that $(X_t)_{t \geqslant 0}$ is a Lévy process?

(b) Let $(N_t^+)_{t \geqslant 0}$ and $(N_t^-)_{t \geqslant 0}$ be independent Poisson processes, each of rate $1/2$, and set $X_t = N_t^+ - N_t^-$. Show that $(X_t)_{t \geqslant 0}$ is a Lévy process and determine the associated characteristic exponent ψ.

(c) Fix $\alpha \in (0,1]$ and define for integers $n \geqslant 1$

$$X_t^{(n)} = n^{-\alpha} X_{nt}.$$

Suppose that the finite-dimensional distributions of $(X_t^{(n)})_{t \geqslant 0}$ converge weakly as $n \to \infty$ to those of another Lévy process $(Y_t)_{t \geqslant 0}$ say. Show that in this case there is only one possible value of α and find the distribution of $(Y_t)_{t \geqslant 0}$.

END OF PAPER