UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 7 June, 2016 9:00 am to 12:00 pm

# PAPER 127

# HOMOTOPY THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

What does it mean to say that a map $\pi : E \to B$ is a *Serre fibration*? Explain how any map is equivalent, in a sense which you should define, to a Serre fibration, and define the *homotopy fibre* of a map.

Describe the long exact sequence of homotopy groups associated to a Serre fibration, defining all of the maps involved. [You do not need to prove that it is exact.]

Let $n \geqslant 2$ and $f : S^1 \vee S^n \to S^n$ be the map which collapses $S^1$ to a point and is the identity on $S^n$. Let $F$ denote the homotopy fibre of $f$. Compute $\pi_n(F)$ as a $\mathbb{Z}[\pi_1(F)]$-module.

**2**

Let $\eta : S^3 \to S^2$ denote the Hopf map. For an integer $d > 0$ let the space $Y_d$ be obtained from $S^2$ by attaching a 4-cell along the map $d \cdot \eta : S^3 \to S^2$, that is, $d$ times the Hopf map in the abelian group $\pi_3(S^2)$ . Compute the ring $H^*(Y_d; \mathbb{Z})$.

Write

$$\eta_1 : S^3 \xrightarrow{\eta} S^2 \subset S^2 \vee S^3 \qquad\qquad i_2 : S^3 \xrightarrow{\mathrm{Id}} S^3 \subset S^2 \vee S^3,$$

and, for integers $d > 0$ and $f$, let the space $X_{d,f}$ be obtained from $S^2 \vee S^3$ by attaching a 4-cell along the map $d \cdot \eta_1 + f \cdot i_2 : S^3 \to S^2 \vee S^3$. Compute the ring $H^*(X_{d,f}; \mathbb{Z})$ and the groups $\pi_1(X_{d,f})$ and $\pi_2(X_{d,f})$.

By considering a map to a suitable Eilenberg–MacLane space, compute $\pi_3(X_{d,f})$.

**3**

Let $G$ be a finite group which acts freely on $S^{n-1}$, for $n \geqslant 3$, such that the induced action on $H^*(S^{n-1}; \mathbb{Z})$ is trivial. Let $\mathbb{F}$ be a field. By constructing a suitable fibration, show that there is an element $\Delta \in H^n(K(G,1); \mathbb{F})$ such that

$$\Delta \smile - : H^i(K(G,1); \mathbb{F}) \longrightarrow H^{i+n}(K(G,1); \mathbb{F})$$

is an isomorphism for all $i > 0$.

Let $p$ be a prime number. By computing $H^*(K(\mathbb{Z}/p, 1); \mathbb{F}_p)$, show that $G$ cannot contain $\mathbb{Z}/p \times \mathbb{Z}/p$ as a subgroup.

**4**

Define the quaternionic projective space $\mathbb{HP}^n$ and compute the ring $H^*(\mathbb{HP}^n;\mathbb{Z})$. By first considering the case $n=2$, compute the map on cohomology induced by the inclusion $i:\mathbb{CP}^n \to \mathbb{HP}^n$.

For an odd prime number $p$, describe the action of the Steenrod operations $\mathcal{P}^i$ on $H^*(\mathbb{HP}^n;\mathbb{F}_p)$. Show that $\mathbb{HP}^{15}/\mathbb{HP}^{12}$ is not homotopy equivalent to $\Sigma^{48}\mathbb{HP}^3$.

## END OF PAPER