UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2016 9:00 am to 11:00 am

PAPER 126

MODULAR FORMS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let f be a meromorphic function on $\mathbb{C}$. What does it mean to say that f is an *elliptic function* with respect to the lattice $\Lambda \subset \mathbb{C}$? Show that an elliptic function with no poles is constant.

Let f be a non-constant elliptic function which is holomorphic on $\mathbb{C} \setminus \Lambda$, with a pole of order $m > 0$ at 0. Show that the sum of the zeroes of f (taken modulo Λ and counted according to multiplicity) is $\equiv 0 \pmod{\Lambda}$, and that the number of zeroes equals m.

Define the Weierstrass $\wp$-function associated to a lattice Λ, and show that it is an elliptic function, holomorphic on $\mathbb{C} \setminus \Lambda$, with a double pole at 0. Show also that for every $a \in \mathbb{C}$, the function $\wp(z) - a$ has either two simple zeroes z, $-z \not\equiv z \pmod{\Lambda}$, or one double zero at a point z with $2z \in \Lambda$.

Show that every elliptic function may be written as $A(\wp(z)) + B(\wp(z))\wp'(z)$ for rational functions A and B.

2

What is *modular form* of weight k (for the full modular group $SL_2(\mathbb{Z})$)? Show that any modular form of weight 0 is constant.

Define the Eisenstein series $G_k(\tau)$ for a positive even integer $k \geqslant 4$, and show that it is a modular form of weight k, with q-expansion

$$G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Let $E_k(\tau) = (2\zeta(k))^{-1} G_k(\tau)$. Show that

$$\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728}$$

is a cusp form of weight 12 whose q-expansion has integral coefficients.

[The formulae $\zeta(4) = \pi^4/2.3^2.5$ and $\zeta(6) = \pi^6/3^3.5.7$ may be helpful.]

3

Write an essay on the theory of Hecke operators for modular forms on $SL_2(\mathbb{Z})$.

4

Let $\Lambda \subset \mathbb{R}^n$ be a lattice. What is the dual lattice $\Lambda' \subset \mathbb{R}^n$? Write down an isomorphism between Λ' and the group of continuous homomorphisms $\mathrm{Hom}(\mathbb{R}^n/\Lambda, \mathbb{C}^\times)$.

Prove the Poisson summation formula

$$\sum_{x \in \Lambda} f(x) = m(\Lambda)^{-1} \sum_{y \in \Lambda'} \hat{f}(y)$$

for a suitably-behaved function f on $\mathbb{R}^n$, where $m(\Lambda)$ is the Euclidean volume of $\mathbb{R}^n/\Lambda$. Hence obtain the functional equation

$$\Theta_\Lambda(\tau) = \sum_{x \in \Lambda} e^{\pi i \|x\|^2 \tau} = (\tau/i)^{-n/2} m(\Lambda)^{-1} \Theta_{\Lambda'}(-1/\tau)$$

for the theta function of Λ.

Deduce that the Epstein zeta function

$$E_\Lambda(s) = \sum_{0 \neq x \in \Lambda} \frac{1}{\|x\|^{2s}}$$

has a meromorphic continuation to $\mathbb{C}$, and satisfies the functional equation

$$\mathcal{E}_\Lambda(s) = \pi^{-s}\Gamma(s)E_\Lambda(s) = m(\Lambda)^{-1}\mathcal{E}_{\Lambda'}(n/2 - s).$$

[Convergence properties of Fourier transforms and series may be assumed without proof, and you may assume that $e^{-\pi x^2}$ is its own Fourier transform.]

5

Let

$$G(\tau, s) = \sum_{m,n}{}' \frac{y^s}{|m\tau + n|^{2s}}$$

be the non-holomorphic Eisenstein series for $SL_2(\mathbb{Z})$, the sum being taken over $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$.

(i) Show that $G(\tau, s)$ is invariant under $SL_2(\mathbb{Z})$, and satisfies the differential equation

$$\Delta G(\tau, s) = s(1-s) G(\tau, s)$$

where

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the Laplace–Beltrami operator on the upper half-plane.

(ii) Show that the constant term $A_0(y, s)$ of the Fourier expansion of $G(\tau, s)$ is given by

$$\pi^{-s} \Gamma(s) A_0(y, s) = 2\xi(2s) y^s + 2\xi(2s-1) y^{1-s}$$

where $\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$.

END OF PAPER