UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 1 June, 2016 9:00 am to 12:00 pm

PAPER 124

PROBABILISTIC NUMBER THEORY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define an *additive function* and a *completely additive function.*

Prove that if f is an additive function, and if $\mathbb{E}_N$ denotes expectation with respect to the discrete uniform probability measure $\mathbb{P}_N$ on $[N] := \{1 \leqslant n \leqslant N\}$, then

$$\mathbb{E}_N f = \sum_{p^k \leqslant N} \frac{f(p^k)}{p^k}\left(1 - \frac{1}{p}\right) + O\left(\frac{1}{N}\sum_{p^k \leqslant N} |f(p^k)|\right).$$

Let $\omega(n)$ and $\Omega(n)$ denote the number of prime factors of n counted without multiplicity and with multiplicity, respectively. Prove that if $t(N)$ is any function that tends to infinity as $N \to \infty$, then

$$\mathbb{P}_N(\Omega - \omega \geqslant t(N)) \to 0 \quad \text{as } N \to \infty.$$

(b) Define the Möbius function $\mu(n)$, and define the Riemann zeta function $\zeta(s)$ for $\Re(s) > 1$.

Prove carefully that when $\Re(s) > 1$ we have

$$\zeta(s) = \prod_{\text{primes } p}\left(1 - \frac{1}{p^s}\right)^{-1}, \qquad \text{and also} \qquad \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1.$$

(c) Suppose we knew that for all small $\epsilon > 0$, we had $\sum_{n \leqslant x} \mu(n) \ll_\epsilon x^{0.9+\epsilon}$ for all x. Deduce the strongest statement you can about the zeros of $\zeta(s)$.

[You may assume that $\zeta(s)$ can be analytically continued to a meromorphic function on $\mathbb{C}$, with only a simple pole at $s = 1$, and you may assume any other basic facts about the zeta function provided you state them clearly.]

2

(a) Prove the First Borel–Cantelli Lemma, which states that if $(A_n)_{n=1}^{\infty}$ is a sequence of events (measurable with respect to a probability measure $\mathbb{P}$), and if $\sum_{n=1}^{\infty} \mathbb{P}(A_n)$ converges, then

$$\mathbb{P}(\text{infinitely many of the } A_n \text{ occur}) =: \mathbb{P}(A_n \ i.o.) = 0.$$

(b) Let $(U_n)_{n=1}^{\infty}$ be as in the Cramér model, so that $U_1 = 0$, and $U_2 = 1$, and $(U_n)_{n \geqslant 3}$ is a sequence of independent Bernoulli random variables taking value 1 with probability $1/\log n$, and taking value 0 otherwise.

For each $n \geqslant 1$, define random variables $P_n := \min\{m : \sum_{i=1}^{m} U_i = n\}$. Prove that

$$\mathbb{P}(\limsup_{n\to\infty} \frac{P_{n+1} - P_n}{\log^2(P_n)} \leqslant 1) = 1.$$

(c) Let $S_n = \sum_{i=1}^{n} X_i$ be a simple random walk, so that X_i are independent random variables taking values ± 1 with probability $1/2$ each. Prove that for any $\theta \in \mathbb{R}$ we have

$$\mathbb{E} \exp\{\theta S_n\} = \left(\frac{e^{\theta} + e^{-\theta}}{2} \right)^n \leqslant e^{n\theta^2/2}.$$

Using this result, together with the fact (*which you may assume without proof*) that for any $\theta > 0$ and $t \in \mathbb{R}$ we have

$$\mathbb{P}(\max_{k \leqslant n} S_k \geqslant t\sqrt{n}) \leqslant e^{-\theta t \sqrt{n}} \mathbb{E} \exp\{\theta S_n\},$$

deduce that for any $t \geqslant 0$ we have

$$\mathbb{P}(\max_{k \leqslant n} S_k \geqslant t\sqrt{n}) \leqslant e^{-t^2/2}.$$

By combining this with part (a), prove that

$$\mathbb{P}(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log \log n}} \leqslant 1) = 1.$$

[Hint: You may find it helpful to consider the special sequence of values $n = \lfloor (1+\delta)^m \rfloor$, for small fixed $\delta > 0$ and $m \in \mathbb{N}$.]

3

(a) Define a z-*sieved* number n, where $z \geqslant 2$.

Define the Buchstab function $w(u)$ for $u \geqslant 1$, and state Buchstab's theorem about $\Phi(z^u, z) := \#\{n \leqslant z^u : n \text{ is } z - \text{sieved}\}$.

(b) *You may assume* that $\lim_{u\to\infty} w(u) = e^{-\gamma}$, where $\gamma \approx 0.577$ is Euler's constant. *You may also assume* that for all $u \geqslant 1$ we have $|w(u) - e^{-\gamma}| \ll \frac{1}{\Gamma(u+1)}$, where $\Gamma(u+1) = \int_0^\infty e^{-x} x^u dx$ is the classical gamma function.

Prove that for any $u \geqslant 1$, there exists a point $u \leqslant u^+ \leqslant u+1$ at which $w(u^+) > e^{-\gamma}$ and there exists a point $u \leqslant u^- \leqslant u+1$ at which $w(u^-) < e^{-\gamma}$.

(c) Let N be large, let $\mathbb{P}_N$ denote the discrete uniform probability measure on $[N]$, and let $\omega(n)$ denote the number of distinct prime factors of n.

State the Turán–Kubilius inequality for the variance of an additive function, and use it to show that

$$\mathbb{P}_N(|\omega(n) - \log\log N| \geqslant 0.01 \log\log N) \ll \frac{1}{\log\log N}.$$

[You may use any standard estimates for sums over primes provided you state them clearly, but you should briefly prove any other estimates that you need.]

Using the above estimate (and NOT using any estimate involving the function $\Omega(n)$), obtain the best upper bound you can for

$$\#\{ab : 1 \leqslant a \leqslant N,\ 1 \leqslant b \leqslant N\}.$$

4

(a) Let $A(s) = \sum_{n \leqslant X} \frac{a_n}{n^s}$ and $B(s) = \sum_{n \leqslant X} \frac{b_n}{n^s}$, where a_n, b_n are arbitrary complex numbers. Prove that for any $T \geqslant 0$ and $\sigma \in \mathbb{R}$, we have

$$\int_T^{2T} A(\sigma+it)\overline{B(\sigma+it)}dt = T\sum_{n \leqslant X} \frac{a_n \overline{b_n}}{n^{2\sigma}} + O\left(\sum_{m,n \leqslant X, m \neq n} \frac{|a_n||b_m|}{n^\sigma m^\sigma |\log(m/n)|}\right).$$

Show that the "big Oh" term here is $\ll X \sum_{m,n \leqslant X, m \neq n} \frac{|a_n||b_m|}{n^\sigma m^\sigma}$, and show also that it is

$$\ll \sum_{n \leqslant X} \frac{|a_n|^2 n \log(2X)}{n^{2\sigma}} + \sum_{m \leqslant X} \frac{|b_m|^2 m \log(2X)}{m^{2\sigma}}.$$

(b) Prove that for any $X \geqslant 2$, any $T \geqslant 0$, any $\sigma \in \mathbb{R}$, and any *odd* $j \in \mathbb{N}$, we have

$$\int_T^{2T} \left(\sum_{\text{primes } p \leqslant X} \frac{\cos(t \log p)}{p^\sigma}\right)^j dt \ll_j X^j \Big(\sum_{n \leqslant X^j} \frac{1}{n^\sigma}\Big)^2.$$

(c) *You may assume* that for all $t \geqslant 1$ we have

$$|\zeta(1/2+it)| \ll \Big|\sum_{n \leqslant \sqrt{t/2\pi}} \frac{1}{n^{1/2+it}}\Big| + \frac{1}{t^{1/4}},$$

where $\zeta(s)$ denotes the Riemann zeta function (a meromorphic function with only a simple pole at $s = 1$).

You may also assume that for all $X \geqslant 2$ we have $\sum_{n \leqslant X} d(n)^2 \ll X(\log X)^3$, where $d(n)$ denotes the total number of divisors of n.

By using these facts, and suitably adapting the argument in part (a), prove that for all $T \geqslant 2$ we have

$$\int_0^T |\zeta(1/2+it)|^4 dt \ll T \sum_{\substack{n_1,\ldots,n_4 \leqslant \sqrt{T/2\pi}, \\ n_1 n_2 = n_3 n_4}} \frac{1}{\sqrt{n_1 n_2 n_3 n_4}} + T(\log T)^4.$$

Deduce that overall we have

$$\int_0^T |\zeta(1/2+it)|^4 dt \ll T(\log T)^4.$$

5

(a) State a general form of the Erdős–Kac central limit theorem.

(b) Suppose $g(n)$ is a strongly additive function (that is, an additive function for which $g(p^k) = g(p)$ for all $k \in \mathbb{N}$). Suppose also that N is large, that $\phi(N) \geqslant 1$ is some function, and that $g(p) = 0$ whenever $p > N^{1/\phi(N)}$.

Furthermore, let $(X_q)_{q \text{ prime}}$ be a sequence of independent random variables (measurable with respect to some probability measure $\mathbb{P}$ with corresponding expectation $\mathbb{E}$), such that X_q takes value 1 with probability $1/q$, and it takes value 0 otherwise.

Prove that if $\mathbb{E}_N$ denotes expectation with respect to the discrete uniform probability measure $\mathbb{P}_N$ on $[N]$, then for any natural number j we have

$$\mathbb{E}_N(g-\mathbb{E}_N g)^j = \sum_{p_1,...,p_j \leqslant N^{1/\phi(N)}} \prod_{i=1}^{w} g(q_i)^{a_i} \mathbb{E}(X_{q_i} - \frac{1}{q_i})^{a_i} + O\Big(\frac{jN^{j/\phi(N)}}{N}\Big(\sum_{p \leqslant N^{1/\phi(N)}} |g(p)|\Big)^j\Big).$$

Here in the sum we write q_i for the *distinct* primes amongst $p_1, ..., p_j$, and a_i for the multiplicities with which they occur, so always $a_1 + ... + a_w = j$.

(c) State a version of the *method of moments* for proving convergence in distribution.

Explain *briefly* how this may be used to prove the Erdős–Kac theorem.

[You should remark briefly on the role of part (b) in the proof, and on the choice of the function $\phi(N)$, but you do NOT need to give any technical details.]

(d) Let $\omega(n)$ denote the number of distinct prime factors of n, and for all large N define

$$E(N) := \sup_{z \in \mathbb{R}} \Big| \mathbb{P}_N(\frac{\omega - \log\log N}{\sqrt{\log\log N}} \leqslant z) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt \Big|.$$

Prove the lower bound

$$E(N) \gg \frac{1}{\sqrt{\log\log N}}.$$

END OF PAPER