TOPICS IN CATEGORY THEORY

Attempt no more than ONE question from Section I
and TWO questions from Section II.

There are SIX questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper
SECTION I

(a) Give a description of the free braided monoidal category $\mathcal{F}br$ on the terminal category $1$, and a description of the free braided strict monoidal category $B$ on the terminal category $1$. *(You are not requested to prove the respective universal properties.)*

(b) Prove that there is a canonical braided strict monoidal functor $\mathcal{F}br \to B$ that is an equivalence. *(You may use the coherence theorem for monoidal categories.)*

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(a) Define the notions of opmonoidal functor and opmonoidal natural transformation.

(b) Define opmonoidal monad. Show that, if a monad on a monoidal category is opmonoidal, then its Eilenberg-Moore category of algebras carries a monoidal structure, and that the associated forgetful functor is strict monoidal.

(c) Deduce that if $H$ is a bialgebra over a commutative ring $k$, prove that the category the left $H$-modules is monoidal and the associated forgetful functor into $k\text{-Mod}$ is strict monoidal.

(d) Continuing with the previous part, prove that if $H$ is Hopf, then the category of left $H$-modules is monoidal left closed.
Let $V$ and $W$ be two monoidal categories. A Frobenius monoidal functor $V \to W$ is a functor $F: V \to W$ equipped with a monoidal structure $(F, \varphi_0, \varphi)$ and an opmonoidal structure $(F, \psi_0, \psi)$ that satisfy the following axioms.

\begin{align*}
FX \otimes F(Y \otimes Z) \xrightarrow{\varphi_{X,Y \otimes Z}} FX \otimes (FY \otimes FZ) &\xrightarrow{\varphi_{X,Y}^{-1}} (FX \otimes FY) \otimes FZ \\
F(X \otimes (Y \otimes Z)) \xrightarrow{F\alpha^{-1}} F((X \otimes Y) \otimes Z) &\xrightarrow{\psi_{X \otimes Y,Z}} F(X \otimes Y) \otimes FZ \\
F(X \otimes Y) \otimes FZ \xrightarrow{\psi_{X,Y,Z}^{-1}} (FX \otimes FY) \otimes FZ &\xrightarrow{\alpha^{-1}} FX \otimes (FY \otimes FZ) \\
F((X \otimes Y) \otimes Z) \xrightarrow{F\alpha} F(X \otimes (Y \otimes Z)) &\xrightarrow{\psi_{X,Y \otimes Z}} FX \otimes F(Y \otimes Z)
\end{align*}

\[(1)\]

(a) Suppose that $e: X \otimes Y \to I$ and $n: I \to Y \otimes X$ are the evaluation and coevaluation of a dual pair in $V$. Prove that a Frobenius structure on the functor $F: V \to W$ as above makes

\[FX \otimes FY \xrightarrow{\varphi_{X,Y}} F(X \otimes Y) \xrightarrow{F\varphi_0} FI \xrightarrow{\psi_0} I \quad I \xrightarrow{\varphi_0} FI \xrightarrow{F\psi_0} F(Y \otimes X) \xrightarrow{\psi_{Y,X}} F(Y \otimes FX).\]

the evaluation and coevaluation of a dual pair.

Let $j: I \to A \leftarrow A \otimes A: m$ be a monoid in the monoidal category $V$. A coseparable structure is a morphism $\varepsilon: A \to I$ such that $\varepsilon \cdot m$ is the evaluation of a dual pair (making $A$ dual to itself). We say that $(A, j, m, \varepsilon)$ is a coseparable monoid.

(b) Prove that if $(A, j, m, \varepsilon)$ is a coseparable monoid in $V$ and $F: V \to W$ is a Frobenius functor, then $FA$ carries a canonical structure of a coseparable monoid.

(c) What can be deduced about the dimension of coseparable monoids in the category of vector spaces over a field (i.e. coseparable algebras)?
A Yang-Baxter operator on an object $X$ in a monoidal category is an invertible endomorphism $y: X \otimes X \to X \otimes X$ such that the following diagram commutes.

\[
\begin{array}{ccccccc}
(X \otimes X) \otimes X & \xrightarrow{\alpha} & X \otimes (X \otimes X) & \xrightarrow{1 \otimes y} & X \otimes (X \otimes X) & \xrightarrow{\alpha^{-1}} & (X \otimes X) \otimes X \\
\downarrow{\gamma \otimes 1} & & \downarrow{\gamma \otimes 1} & & \downarrow{\gamma \otimes 1} & & \downarrow{\gamma \otimes 1} \\
X \otimes (X \otimes X) & \xrightarrow{1 \otimes y} & X \otimes (X \otimes X) & \xrightarrow{\alpha} & X \otimes (X \otimes X) & \xrightarrow{1 \otimes y} & (X \otimes X) \otimes X \\
\end{array}
\]

(1)

(a) (i) Show that each object in a braided monoidal category carries a canonical Yang-Baxter operator.

(ii) Show that each Yang-Baxter operator $y: X \otimes X \to X \otimes X$ in a strict monoidal category $C$ induces a strict monoidal functor $B \to C$ that sends $1 \in B$ to $X$; here $B$ is the braid category.

(b) Let $H$ be a bimonoid in a symmetric monoidal category $V$ equipped with a coquasitriangular structure $\gamma: H \otimes H \to I$. Prove that $\gamma$ satisfies

\[
(H^\otimes 3 \xrightarrow{\delta \otimes \delta \otimes \delta} H^\otimes 6 \xrightarrow{1 \otimes \delta \otimes 1 \otimes 1} H^\otimes 6 \xrightarrow{\gamma \otimes \gamma \otimes \gamma} I) =

(H^\otimes 3 \xrightarrow{\delta \otimes \delta \otimes \delta} H^\otimes 6 \xrightarrow{1 \otimes 1 \otimes 1 \otimes 1} H^\otimes 6 \xrightarrow{1 \otimes \gamma \otimes \gamma \otimes 1} H^\otimes 2 \xrightarrow{\gamma} I),
\]

where $c$ denotes the symmetry of $V$. [Hint: You may wish to use the part (a)(i).]
(The last part of this problem is independent of the rest.)

(a) Let \( f : H \rightarrow K \) be a morphism of comonoids between bimonoids in a braided monoidal category. Show that the induced functor \( f^* : \text{Comod}(H) \rightarrow \text{Comod}(K) \) is strict monoidal if and only if \( f \) is a morphism of monoids.

(b) Let \( f, g : H \rightarrow K \) be two morphisms of bialgebras over a field \( k \) and assume that \( H \) is a Hopf algebra. If \( f^*, g^* : \text{Comod}(H) \rightarrow \text{Comod}(K) \) are the induced functors between their categories of comodules, prove that any monoidal natural transformation \( \beta : f^* \Rightarrow g^* \) is invertible. (Hint: you may wish to consider a certain map \( H \rightarrow k \) related to \( \beta \).)

(c) Let \((H, j, m, \delta, \varepsilon, S)\) be a Hopf algebra in vector spaces over a field \( k \), with antipode \( S : H \rightarrow H \). If \( M \) is a left \( H \)-module, define its subspace of invariants as

\[
M^H = \{ m \in M : x \cdot m = \varepsilon(x)m \ \forall x \in H \}.
\]

Regard \( H \) as a left \( H \)-module via its multiplication, and endow \( \text{Hom}_k(H, H) \) with its induced left \( H \)-module structure. Prove that \((\text{Hom}_k(H, H))^H\) is the subspace of morphisms of left \( H \)-modules \( H \rightarrow H \).

(a) Let \( U : \mathcal{C} \rightarrow \mathcal{V} \) be a faithful functor into a monoidal category \( \mathcal{V} \). Assume that \( \mathcal{C} \) is equipped with two monoidal structures \((\mathcal{C}, I, \bullet)\) and \((\mathcal{C}, I, \cdot)\) that share the unit object \( I \), and that make \( U \) a strict monoidal functor. What conditions on a natural transformation \( \varphi_{X,Y} : X \bullet Y \rightarrow X \cdot Y \) make \( \varphi \) together with the identity morphism \( 1 = I \rightarrow I \) into a monoidal structure for the identity functor \((\mathcal{C}, I, \cdot) \rightarrow (\mathcal{C}, I, \bullet)\)? You should express these conditions as commutative diagrams.

(b) Suppose that a coalgebra \( H \) over a field \( k \) has a unit \( j \) and two multiplications \( m \) and \( n \) that make \((H, j, m)\) and \((H, j, n)\) into bimonoids, giving rise to two tensor products \( \circ, \bullet : \text{Comod}(H)^2 \rightarrow \text{Comod}(H) \). Consider monoidal structures \((\varphi, \varphi_0)\) on the identity functor

\[
1 : (\text{Comod}(H), k, \circ) \rightarrow (\text{Comod}(H), k, \bullet)
\]

such that \( \varphi_0 : k \rightarrow k \) is the identity morphism. Classify these monoidal structures \( \phi \) in terms of linear maps \( H \otimes H \rightarrow k \). (Hint: you may wish to use part (a).)
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