UNIVERSITY OF CAMBRIDGE

## MATHEMATICAL TRIPOS Part III

Wednesday, 1 June, 2016 1:30 pm to 4:30 pm

# PAPER 122

# TOPICS IN CATEGORY THEORY

*Attempt no more than* ***ONE*** *question from Section I*
*and* ***TWO*** *questions from Section II.*

*There are* ***SIX*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

## SECTION I

**1**

(a) Give a description of the free braided monoidal category $\mathcal{F}br$ on the terminal category $\mathbf{1}$, and a description of the free braided strict monoidal category $\mathbf{B}$ on the terminal category $\mathbf{1}$. *(You are not requested to prove the respective universal properties.)*

(b) Prove that there is a canonical braided strict monoidal functor $\mathcal{F}br \to \mathbf{B}$ that is an equivalence. *(You may use the coherence theorem for monoidal categories.)*

**2**

(a) Define the notions of *opmonoidal functor* and *opmonoidal natural transformation.*

(b) Define *opmonodial monad.* Show that, if a monad on a monoidal category is opmonoidal, then its Eilenberg-Moore category of algebras carries a monoidal structure, and that the associated forgetful functor is strict monoidal.

(c) Deduce that if $H$ is a bialgebra over a commutative ring $k$, prove that the category the left $H$-modules is monoidal and the associated forgetful functor into $k$-**Mod** is strict monoidal.

(d) Continuing with the previous part, prove that if $H$ is Hopf, then the category of left $H$-modules is monoidal left closed.

## SECTION II

**3**

Let $\mathcal{V}$ and $\mathcal{W}$ be two monoidal categories. A *Frobenius monoidal functor* $\mathcal{V} \to \mathcal{W}$ is a functor $F\colon \mathcal{V} \to \mathcal{W}$ equipped with a monoidal structure $(F, \varphi_0, \varphi)$ and an opmonoidal structure $(F, \psi_0, \psi)$ that satisfy the following axioms.

$$\begin{CD}
FX \otimes F(Y \otimes Z) @>{1\otimes\psi_{Y,Z}}>> FX \otimes (FY \otimes FZ) @>{\alpha^{-1}}>> (FX \otimes FY) \otimes FZ \\
@V{\varphi_{X,Y\otimes Z}}VV @. @VV{\varphi_{X,Y}\otimes 1}V \\
F(X \otimes (Y \otimes Z)) @>{F\alpha^{-1}}>> F((X \otimes Y) \otimes Z) @>{\psi_{X\otimes Y,Z}}>> F(X \otimes Y) \otimes FZ
\end{CD} \tag{1}$$

$$\begin{CD}
F(X \otimes Y) \otimes FZ @>{\psi_{X,Y}\otimes 1}>> (FX \otimes FY) \otimes FZ @>{\alpha}>> FX \otimes (FY \otimes FZ) \\
@V{\varphi_{X\otimes Y,Z}}VV @. @VV{1\otimes\varphi_{Y,Z}}V \\
F((X \otimes Y) \otimes Z) @>{F\alpha}>> F(X \otimes (Y \otimes Z)) @>{\psi_{X,Y\otimes Z}}>> FX \otimes F(Y \otimes Z)
\end{CD} \tag{2}$$

(a) Suppose that $e\colon X \otimes Y \to I$ and $n\colon I \to Y \otimes X$ are the evaluation and coevaluation of a dual pair in $\mathcal{V}$. Prove that a Frobenius structure on the functor $F\colon \mathcal{V} \to \mathcal{W}$ as above makes

$$FX{\otimes}FY \xrightarrow{\varphi_{X,Y}} F(X{\otimes}Y) \xrightarrow{Fe} FI \xrightarrow{\psi_0} I \qquad I \xrightarrow{\varphi_0} FI \xrightarrow{Fn} F(Y{\otimes}X) \xrightarrow{\psi_{Y,X}} FY{\otimes}FX.$$

the *evaluation and coevaluation of a dual pair.*

Let $j\colon I \to A \leftarrow A \otimes A\colon m$ be a monoid in the monoidal category $\mathcal{V}$. A *coseparable* structure is a morphism $\varepsilon\colon A \to I$ such that $\varepsilon \cdot m$ is the evaluation of a dual pair (making $A$ dual to itself). We say that $(A, j, m, \varepsilon)$ is a coseparable monoid.

(b) Prove that if $(A, j, m, \varepsilon)$ is a coseparable monoid in $\mathcal{V}$ and $F\colon \mathcal{V} \to \mathcal{W}$ is a Frobenius functor, then $FA$ carries a canonical structure of a coseparable monoid.

(c) What can be deduced about the dimension of coseparable monoids in the category of vector spaces over a field (i.e. coseparable algebras)?

**4**

A Yang-Baxter operator on an object $X$ in a monoidal category is an invertible endomorphism $y\colon X \otimes X \to X \otimes X$ such that the following diagram commutes.

$$\begin{array}{ccccccc}
(X \otimes X) \otimes X & \xrightarrow{\alpha} & X \otimes (X \otimes X) & \xrightarrow{1\otimes y} & X \otimes (X \otimes X) & \xrightarrow{\alpha^{-1}} & (X \otimes X) \otimes X \\
{\scriptstyle y\otimes 1}\Big\uparrow & & & & & & \Big\downarrow{\scriptstyle y\otimes 1} \\
(X \otimes X) \otimes X & & & & & & (X \otimes X) \otimes X \\
{\scriptstyle \alpha}\Big\downarrow & & & & & & \Big\downarrow{\scriptstyle \alpha} \\
X \otimes (X \otimes X) & & & & & & X \otimes (X \otimes X) \\
\Big\downarrow{\scriptstyle 1\otimes y} & & & & & & {\scriptstyle 1\otimes y}\Big\uparrow \\
X \otimes (X \otimes X) & \xrightarrow{\alpha^{-1}} & (X \otimes X) \otimes X & \xrightarrow{y\otimes 1} & (X \otimes X) \otimes X & \xrightarrow{\alpha} & X \otimes (X \otimes X)
\end{array} \tag{1}$$

(a) (i) Show that each object in a braided monoidal category carries a canonical Yang-Baxter operator.

(ii) Show that each Yang-Baxter operator $y\colon X \otimes X \to X \otimes X$ in a strict monoidal category $\mathcal{C}$ induces a *strict monoidal functor* $\mathbf{B} \to \mathcal{C}$ that sends $1 \in \mathbf{B}$ to $X$; here $\mathbf{B}$ is the braid category.

(b) Let $H$ be a bimonoid in a symmetric monoidal category $\mathcal{V}$ equipped with a coquasi-triangular structure $\gamma\colon H \otimes H \to I$. Prove that $\gamma$ satisfies

$$\big(H^{\otimes 3} \xrightarrow{\delta\otimes\delta\otimes\delta} H^{\otimes 6} \xrightarrow{1\otimes c\otimes c\otimes 1} H^{\otimes 6} \xrightarrow{\gamma\otimes\gamma\otimes\gamma} I\big) =$$
$$= \big(H^{\otimes 3} \xrightarrow{\delta\otimes\delta\otimes\delta} H^{\otimes 6} \xrightarrow{1\otimes 1\otimes c\otimes 1\otimes 1} H^{\otimes 6} \xrightarrow{1\otimes\gamma\otimes\gamma\otimes 1} H^{\otimes 2} \xrightarrow{\gamma} I\big),$$

where $c$ denotes the symmetry of $\mathcal{V}$. [*Hint: You may wish to use the part (a)(i).*]

**5**

*(The last part of this problem is independent of the rest.)*

(a) Let $f\colon H \to K$ be a morphism of comonoids between bimonoids in a braided monoidal category. Show that the induced functor $f_*\colon \mathbf{Comod}(H) \to \mathbf{Comod}(K)$ is strict monoidal if and only if $f$ is a morphism of monoids.

(b) Let $f, g\colon H \to K$ be two morphisms of bialgebras over a field $k$ and assume that $H$ is a Hopf algebra. If $f_*, g_*\colon \mathbf{Comod}(H) \to \mathbf{Comod}(K)$ are the induced functors between their categories of comodules, prove that any monoidal natural transformation $\beta\colon f_* \Rightarrow g_*$ is invertible. *(Hint: you may wish to consider a certain map $H \to k$ related to $\beta$.)*

(c) Let $(H, j, m, \delta, \varepsilon, S)$ be a Hopf algebra in vector spaces over a field $k$, with antipode $S\colon H \to H$. If $M$ is a left $H$-module, define its *subspace of invariants* as

$$M^H = \{m \in M : x \cdot m = \varepsilon(x)m \ \forall x \in H\}.$$

Regard $H$ as a left $H$-module via its multiplication, and endow $\mathrm{Hom}_k(H, H)$ with its induced left $H$-module structure. Prove that $(\mathrm{Hom}_k(H, H))^H$ is the subspace of morphisms of left $H$-modules $H \to H$.

**6**

(a) Let $U\colon \mathcal{C} \to \mathcal{V}$ be a faithful functor into a monoidal category $\mathcal{V}$. Assume that $\mathcal{C}$ is equipped with two monoidal structures $(\mathcal{C}, I, \bullet)$ and $(\mathcal{C}, I, \diamond)$ that share the unit object $I$, and that make $U$ a strict monoidal functor. What conditions on a natural transformation $\varphi_{X,Y}\colon X \bullet Y \to X \diamond Y$ make $\varphi$ together with the identity morphism $1 = I \to I$ into a monoidal structure for the identity functor $(\mathcal{C}, I, \diamond) \to (\mathcal{C}, I, \bullet)$? You should express these conditions as commutative diagrams.

(b) Suppose that a coalgebra $H$ over a field $k$ has a unit $j$ and two multiplications $m$ and $n$ that make $(H, j, m)$ and $(H, j, n)$ into bimonoids, giving rise to two tensor products $\diamond, \bullet\colon \mathbf{Comod}(H)^2 \to \mathbf{Comod}(H)$. Consider monoidal structures $(\varphi, \varphi_0)$ on the identity functor

$$1\colon (\mathbf{Comod}(H), k, \diamond) \longrightarrow (\mathbf{Comod}(H), k, \bullet)$$

such that $\varphi_0\colon k \to k$ is the identity morphism. Classify these monoidal structures $\phi$ in terms of linear maps $H \otimes H \to k$. *(Hint: you may wish to use part (a).)*

**END OF PAPER**