UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 1 June, 2016 9:00 am to 12:00 pm

PAPER 121

TOPICS IN SET THEORY

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define the gimel function $\gimel$ and give a precise statement of the singular cardinals hypothesis (SCH).

(b) Assuming $\mathrm{cf}(2^{\aleph_0}) = \aleph_1 < 2^{\aleph_0}$, show that there is an uncountable set $\mathcal{A}$ of infinite sets of reals numbers of pairwise different size (i.e., if $A, B \in \mathcal{A}$, then $|A| \neq |B|$) such that none of them has size $2^{\aleph_0}$ and $\bigcup \mathcal{A} = \mathbb{R}$.

In the following, let σ_α be the αth uncountable singular cardinal, i.e., $\sigma_0 = \aleph_\omega$, $\sigma_1 = \aleph_{\omega+\omega}$, and so on.

(c) Determine the least α such that σ_α does not have countable cofinality. (Justify your answer.)

(d) Is there a cardinal κ such that $\sigma_\kappa = \kappa$? (Justify your answer.)

(e) We write (σ) for the statement "there is a limit ordinal α such that $\mathrm{cf}(\sigma_\alpha) \neq \mathrm{cf}(\alpha)$". Show that if ZFC is consistent, then ZFC does not prove (σ).

2

(a) Give a precise statement of Freiling's axiom of symmetry $\mathsf{A}_{<\omega_1}(\mathbb{R})$ and the generalised continuum hypothesis GCH.

(b) Prove Freiling's Theorem: Freiling's axiom of symmetry is equivalent to the negation of the continuum hypothesis.

(c) A set is called meagre if it is a countable union of nowhere dense sets. An uncountable set $A \subseteq \mathbb{R}$ is called *Lusin* if for all meagre sets $M \subseteq \mathbb{R}$, the set $A \cap M$ is countable. Prove that CH implies the existence of a Lusin set.

(d) We call a set X a *linear increasing union of countable sets* if there is a linear order $(I, \leqslant)$ and a family of countable sets $\{X_i\,;\, i \in I\}$ such that if $i \leqslant j$, then $X_i \subseteq X_j$, and furthermore $\bigcup_{i\in I} X_i = X$.

Prove that a set that is a linear increasing union of countable sets has at most cardinality $\aleph_1$.

(e) Suppose that there is a Lusin set of cardinality $2^{\aleph_0}$ and that every set of cardinality $< 2^{\aleph_0}$ is meagre. Under these hypotheses, prove that CH holds.

3

(a) Define the notions of a club set, the club filter, an ω-measurable cardinal, and a measurable cardinal.

(b) Show that for arbitrary regular uncountable cardinals κ, the club filter on κ contains sets that are stationary but not club.

(c) Show that the club filter on $\aleph_2$ is $\aleph_2$-complete, but not an ultrafilter.

(d) A weakly inaccessible cardinal κ is called *weakly Mahlo* if $\{\alpha < \kappa\,;\, \alpha = \mathrm{cf}(\alpha)\}$ is stationary in κ. Show that if κ is weakly Mahlo, then there are unboundedly many weakly inaccessible cardinals below κ.

(e) Prove that if there is an ω-measurable cardinal then there is a measurable cardinal.

4

(a) Define both the von Neumann hierarchy $\mathbf{V}_\alpha$ and the constructible hierarchy $\mathbf{L}_\alpha$ for ordinals α by transfinite recursion. Define the set $\mathbf{H}_\kappa$ of sets of hereditary cardinality $< \kappa$.

(b) Prove that there is an ordinal α such that $\mathbf{L} \cap \mathbf{V}_\alpha \neq \mathbf{L}_\alpha$.

(c) Show that the following properties are expressible by Π_1^{ZF}-formulae:

(i) κ is a cardinal,

(ii) κ is regular.

(d) Prove that $\mathbf{H}_{\omega_2} \not\models \mathsf{PowerSet}$.

(e) A cardinal ι is called *inaccessible limit of inaccessibles* if it is inaccessible and the set of inaccessibles below it is unbounded in ι. Let $\infty\mathsf{IC}$ be the statement "there are unboundedly many inaccessible cardinals" and ILI be the statement "there is an inaccessible limit of inaccessibles". Show that if $\mathsf{ZFC} + \infty\mathsf{IC}$ is consistent, then it does not prove ILI.

5

In all of the following, M is a countable transitive model of ZFC, $\mathbb{P}$ is a partial order in M, and λ is a cardinal in M.

(a) Define the notions of "$\mathbb{P}$ is λ-closed" and "$\mathbb{P}$ has the λ-chain condition".

(b) Assuming that $\mathbb{P}$ is a separative partial order in M, show that there is a G that is $\mathbb{P}$-generic over M and $G \notin M$.

(c) Suppose that τ and τ' are $\mathbb{P}$-names. Consider the $\mathbb{P}$-name

$$\nu(\tau, \tau') := \{(p, \tau)\,;\, p \in \mathbb{P}\} \cup \{(q, \tau')\,;\, q \in \mathbb{P}\}.$$

Let G be a $\mathbb{P}$-generic filter over M. Determine $(\nu(\tau, \tau'))_G$ and describe in words what this set is.

(d) Assuming that $\mathbb{P}$ is a λ-closed partial order and that G is $\mathbb{P}$-generic over M, show that $M[G] \models$"λ is a cardinal".

(e) Let α_1 and $\alpha_2 \in M$ be such that $M \models \aleph_1 = \alpha_1 \wedge \aleph_2 = \alpha_2$. Define (in M) the partial order $\mathbb{P}$ consisting of functions $p : \mathrm{dom}(p) \to 2$ where $\mathrm{dom}(p)$ is a finite subset of $\alpha_2 \times \omega$, ordered by inclusion. Suppose that G is $\mathbb{P}$-generic over M.

  (i) Show that $M[G] \models$"there is an injection from α_2 into the power set of ω".

  (ii) Show that $M[G] \models$"α_1 is a cardinal".

END OF PAPER