UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 26 May, 2016 9:00 am to 12:00 pm

PAPER 120

COMPUTABILITY AND LOGIC

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let G be a group, and (A, L) an automatic structure for G. Show that there exists a constant N such that, if $w \in L$ and $g \in G$ satisfy $g = \overline{wx}$ or $\overline{w} = g\overline{x}$ for some $x \in A \cup \{\epsilon\}$, then:
(i) g has some representative $v \in L$ of length $|v| \leqslant |w| + N$; and
(ii) If $u \in L$ is a representative of g with $|u| > |w| + N$, then there are infinitely many representatives of g in L.

Henceforth let G be a group, and (A, L) an automatic structure for G with A a symmetric set (that is, if $a \in A$, then $a^{-1} \in A$). Assume that all multiplier automata M_x $(x \in A \cup \{\epsilon\})$ are normalised; they have no inaccessible states, and all dead states are merged into one.

(b) Show that, given $u \in L$ and $x \in A$, we can algorithmically construct $v \in L$ with $\overline{v} = \overline{ux}$ in G.

(c) Using the result of part (a), show that the algorithm you described to compute v in part(b) can be carried out in time $O(|u|)$, and moreover that v can always be constructed such that $|v| \leqslant |u| + N$ for some fixed N which depends only on the automatic structure (A, L).

(d) Show that, from (A, L), we can construct a word $\gamma \in L$ such that $\overline{\gamma} = 1$ in G.

(e) Let γ be as in part (d). Given a word $w \in A^*$, show that we can construct a word $z \in L$ with $\overline{z} = \overline{w}$ in G, in time $O(|w|^2)$, with $|z| \leqslant |w|N + |\gamma|$.

(f) Conclude that there is an algorithm that, on input of any word $w \in A^*$ decides, in time $O(|w|^2)$, whether or not $\overline{w} = 1$ in G.

2

Let A be an infinite set, and let G be a subset of A^ω, the set of ω-sequences of elements of A. In the game G_A players I (who plays first) and II alternately pick members from A (with replacement, so repetitions are allowed) thereby generating a *play* $p \in A^\omega$. I wins iff $p \in G$. Give A the discrete topology, A^ω the product topology. By using a fixed-point theorem or otherwise show that if $G \subseteq A^\omega$ is open then one of the two players must have a winning strategy.

[If you wish to appeal to a fixed-point theorem you should state it correctly.]

3

State and prove Rice's theorem. If you wish to appeal to other theorems you must state them correctly but need not prove them.

4

(1) (a) What is a primitive recursive function?

(b) Define by recursion an ω-sequence of functions $\mathbb{N}^2 \to \mathbb{N}$ of which the first three members are addition, multiplication and exponentiation.

(c) Prove that every function in your sequence is primitive recursive.

(2) Is every semidecidable set $X \subseteq \mathbb{N}$ the range of a primitive recursive $f : \mathbb{N} \to \mathbb{N}$?

5

What are *many-one reducibility* and *Turing-reducibility*?

State and prove the Friedberg-Muchnik theorem.

6

Explain how every computable function can be represented by a λ-term.

END OF PAPER