UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 31 May, 2016 9:00 am to 12:00 pm

PAPER 119

INTRODUCTION TO CATEGORY THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the terms *monomorphism*, *strong* (or *extremal*) *monomorphism* and *regular monomorphism*. Show that any regular monomorphism is strong, and that if two subobjects $A' \rightarrowtail A$, $A'' \rightarrowtail A$ are strong then so is their intersection (= pullback) $A' \cap A'' \rightarrowtail A$, if it exists.

We call a morphism *anodyne* if it is both monic and epic, and we call an object B *saturated* if it is injective with respect to the class of anodyne morphisms, i.e. every diagram

$$\begin{array}{ccc} A' & \rightarrowtail\!\!\!\twoheadrightarrow & A \\ \downarrow & & \\ B & & \end{array}$$

can be completed to a commutative triangle. Show that a strong subobject of a saturated object is saturated.

Now suppose that $\mathcal{C}$ is complete and well-powered, and that every object A of $\mathcal{C}$ admits a monomorphism $A \rightarrowtail B$ with B saturated. Show that every object admits an anodyne morphism to a saturated object [*hint: consider the smallest strong subobject of B which contains A*]. Deduce that the full subcategory $\mathcal{S}$ of saturated objects is reflective in $\mathcal{C}$. Show also that $\mathcal{S}$ is balanced [*hint: first show that epimorphisms in $\mathcal{S}$ are also epic in $\mathcal{C}$*].

2 State the Yoneda Lemma. If $\mathcal{C}$ is a small category, show that any functor $F : \mathcal{C} \to \mathbf{Set}$ may be expressed as a colimit in $[\mathcal{C}, \mathbf{Set}]$ of a diagram of shape $(1 \downarrow F)^{\mathrm{op}}$ whose vertices are representable functors, where 1 denotes a singleton set. [*Hint: observe that a natural transformation $\alpha : F \to G$ is determined by the family of elements $(\alpha_A(x) \mid A \in \mathrm{ob}\, \mathcal{C}, x \in FA)$.*]

Now suppose $\mathcal{C}$ has finite limits. Show that the following conditions on F are equivalent:

(i) F preserves finite limits.

(ii) For any set S, $(S \downarrow F)$ has finite limits.

(iii) $(1 \downarrow F)^{\mathrm{op}}$ is filtered.

(iv) F is expressible as a filtered colimit of representable functors.

[You may assume the result that filtered colimits commute with finite limits in **Set**.]

3 Let $\mathcal{C}$ be a category with finite products. An object A of $\mathcal{C}$ is said to be *exponentiable* if the functor $(-)\times A\colon \mathcal{C} \to \mathcal{C}$ has a right adjoint $(-)^A$. Show that the class of exponentiable objects of $\mathcal{C}$ is closed under finite products.

Now suppose that $\mathcal{C}$ is complete and locally small, that it has a coseparator S, and that every monomorphism in $\mathcal{C}$ is regular. Show that for every object B of $\mathcal{C}$ there is an equalizer diagram of the form

$$B \rightarrowtail \prod_{i\in I} S \rightrightarrows \prod_{j\in J} S$$

for suitable index sets I and J. Deduce that an object A is exponentiable if and only if the functor $\mathcal{C}\,(-\times A, S)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable. [*Hint: if the functor* $(-)^A$ *exists, it preserves limits.*]

4 Explain what is meant by the *monadic length* of an adjunction

$$(F\colon \mathcal{C} \to \mathcal{D} \dashv G\colon \mathcal{D} \to \mathcal{C}) \ ,$$

where $\mathcal{D}$ has reflexive coequalizers. [You should write down the definitions of the Eilenberg–Moore comparison functor and its left adjoint, but you need not verify their properties.]

For each natural number n, let $\mathcal{C}_n$ denote the category whose objects are sets A equipped with n partial unary operations $\alpha_1, \alpha_2 \ldots, \alpha_n$, such that $\alpha_1(a)$ is defined for all $a \in A$, and for $i > 1$ $\alpha_i(a)$ is defined if and only if ($\alpha_{i-1}(a)$ is defined and) $\alpha_{i-1}(a) = a$, and whose morphisms $(A, \alpha_1, \ldots, \alpha_n) \to (B, \beta_1, \ldots, \beta_n)$ are functions $f : A \to B$ such that $f(\alpha_i(a)) = \beta_i(f(a))$ whenever $\alpha_i(a)$ is defined, for each i. Show that the forgetful functor $G_n\colon \mathcal{C}_{n+1} \to \mathcal{C}_n$ has a left adjoint F_n [*hint:* $F_n(A, \alpha_1, \ldots, \alpha_n)$ *may be taken to have underlying set* $(A \times \{0\}) \cup (A_n \times \mathbb{N})$, *where* $A_n = \{a \in A \mid \alpha_n(a) = a\}$.]

Show also that, for each $m > n$, the composite adjunction

$$\mathcal{C}_n \underset{G_n}{\overset{F_n}{\rightleftarrows}} \mathcal{C}_{n+1} \underset{G_{n+1}}{\overset{F_{n+1}}{\rightleftarrows}} \mathcal{C}_{n+2} \quad \cdots \quad \mathcal{C}_{m-1} \underset{G_{m-1}}{\overset{F_{m-1}}{\rightleftarrows}} \mathcal{C}_m$$

induces the same monad (up to isomorphism) on $\mathcal{C}_n$ as $\mathcal{C}_n \rightleftarrows \mathcal{C}_{n+1}$. Show that G_n creates coequalizers of G_n-split pairs, and deduce that the composite adjunction displayed above has monadic length $m-n$.

5 Explain carefully what is meant by the assertion that limits of shape I commute with colimits of shape J in a category $\mathcal{C}$.

Let G be a group, considered as a category with one object, and let A be a G-set, regarded as a functor $G \to \mathbf{Set}$. Show that $\lim_G A$ and $\mathrm{colim}_G A$ may be identified respectively with the set of G-fixed elements of A and the set of G-orbits.

Hence show that

(a) if G and H are finite groups of coprime orders, then limits of shape G commute with colimits of shape H in $\mathbf{Set}$;

(b) if G and H have a nontrivial common quotient group K, then limits of shape G do not commute with colimits of shape H in $\mathbf{Set}$.

[*Hint for* (b)*: let* G *and* H *act on (the underlying set of)* K *by* $(g, k) \mapsto q(g).k$ *and* $(h, k) \mapsto k.r(h)^{-1}$*, where* q *and* r *are the quotient maps.*]

6 Define the notion of *abelian category*, and prove from your definition that epimorphisms in an abelian category are stable under pullback. [Standard results on additive categories may be assumed.]

Now suppose given an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ and a pullback square

$$\begin{array}{ccc} B' & \xrightarrow{g'} & C' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ B & \xrightarrow{g} & C \end{array}$$

in an abelian category. Show that there is a morphism $f' \colon A \to B'$ making the sequence $0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$ exact.

Deduce that if $0 \to K \to P \to C \to 0$ and $0 \to K' \to P' \to C \to 0$ are exact sequences with P and P' projective, then $K \oplus P' \cong K' \oplus P$. [*Hint: show that they are both isomorphic to the pullback of* $P \to C$ *and* $P' \to C$.]

END OF PAPER