UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS **Part III**

Friday, 3 June, 2016 9:00 am to 12:00 pm *Draft 25 July, 2016*

PAPER 118

COMPLEX MANIFOLDS

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let M be a complex manifold. Define the space $\mathcal{A}^{p,q}(M)$ of forms of type (p,q) on M. Define the operators $\partial : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p+1,q}(M)$ and $\bar{\partial} : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q+1}(M)$.

Given a holomorphic map $f : M \to N$, show that f induces a pull-back map $f^* : \mathcal{A}^{p,q}(N) \to \mathcal{A}^{p,q}(M)$.

(b) Let $D = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, |z_i| < 1\}$. Show that if α is a (p,q)-form on D with $d\alpha = 0$, then there is a form β which is a sum of forms of type $(p-1,q)$ and $(p,q-1)$ with $d\beta = \alpha$.

(c) Let $\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ be the standard Kähler form on $\mathbb{C}^2$. Find a different Kähler form ω' on $\mathbb{C}^2 - \{(0,0)\}$ such that the two corresponding metrics have the same volume form. [*Hint: Look for* $\omega' = i\partial\bar{\partial}\varphi(|z_1|^2 + |z_2|^2)$*, where* $\varphi : \mathbb{R}_{>0} \to \mathbb{R}$ *is a function.*]

2

(a) Show that on a Kähler manifold, holomorphic functions are harmonic with respect to Δ_d.

(b) On a Riemannian manifold M with metric g_{ij}, the Laplacian Δ_d on functions (i.e., 0-forms) takes the form

$$\Delta_d(u) = \sum_{k,l=1}^{n} \frac{1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x_k} \left(\sqrt{\det(g_{ij})} g^{kl} \frac{\partial u}{\partial x_l} \right)$$

where $(g^{ij}) = (g_{ij})^{-1}$. Show that if M is in fact a complex manifold of dimension n, with local complex coordinates $z_j = x_j + \sqrt{-1}x_{n+j}$, and g is a Kähler metric, then this simplifies to

$$\Delta_d(u) = \sum_{k,l=1}^{2n} g^{kl} \frac{\partial^2 u}{\partial x_k \partial x_l}.$$

(c) Let $X_1 = \mathbb{C}^{g_1}/\Lambda_1$, $X_2 = \mathbb{C}^{g_2}/\Lambda_2$ be complex tori. Let $f : X_1 \to X_2$ be a holomorphic map. Show that there exists an $x \in \mathbb{C}^{g_2}$ and a linear transformation $\tilde{f} : \mathbb{C}^{g_1} \to \mathbb{C}^{g_2}$ with $\tilde{f}(\Lambda_1) \subseteq \Lambda_2$ such that $f(z + \Lambda_1) = \tilde{f}(z) + x + \Lambda_2$.

(d) Show that if $X = V/\Lambda$ is a complex torus, then there are isomorphisms

$$H^{p,q}_{\bar{\partial}}(X) \cong \bigwedge^p \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \otimes \bigwedge^q \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$$

where

$$\mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C}) = \{f : V \to \mathbb{C} | f \ \mathbb{R}\text{-linear and } f(cv) = \bar{c}f(v) \text{ for all } c \in \mathbb{C}, v \in V\}$$

denotes the space of $\mathbb{C}$-antilinear homomorphisms.

3

(a) Let E be a vector bundle. Define the notion of a *connection* D on E, and define the *curvature* Θ of D.

If E_1, E_2 are two vector bundles with connections D_1, D_2, construct a connection on $E_1 \otimes E_2$ from D_1, D_2, and prove that the curvature Θ of this connection satisfies

$$\Theta = \Theta_1 \otimes 1 + 1 \otimes \Theta_2.$$

(b) Given a Hermitian metric h on a holomorphic vector bundle E over a complex manifold M, prove there is a unique connection D on E compatible with both the metric and holomorphic structure on E. Describe this connection in terms of a holomorphic frame as a matrix of 1-forms.

(c) Let h be a Hermitian metric on a holomorphic vector bundle E over a complex manifold M. Let $S \subseteq E$ be a holomorphic sub-bundle. Then S inherits a Hermitian structure from E. In particular, E and S carry connections D_E and D_S given by (b).

Show that the quotient bundle $Q = E/S$ can be naturally identified with $S^{\perp} \subseteq E$ as C^{∞} vector bundles, and thus Q inherits a Hermitian structure.

Let $\pi_S : E \to S$ denote the orthogonal projection, inducing also $\pi_S : T^*_M \otimes E \to T^*_M \otimes S$. Show that $D_S = \pi_S \circ D_E$.

Define the operator A on C^{∞} sections of S by

$$A(s) = D_E(s) - D_S(s).$$

Show that $A(s)$ is a C^{∞} section of $T_M \otimes Q$ (viewing Q as a subbundle of E) and that

$$A(fs) = fA(s)$$

for f a C^{∞} function on M.

4

(a) Let X be a topological space and $\mathfrak{U}$ an open cover of X. Let $\mathcal{F}$ be a sheaf on X. Define the Čech cohomology group $\check{H}^q(\mathfrak{U}, \mathcal{F})$.

(b) Let $X = \mathbb{C}^2 \setminus \{(0,0)\}$, $\mathfrak{U} = \{U_1, U_2\}$ the open cover given by

$$U_1 = X \setminus \{(z,0) \mid z \in \mathbb{C} \setminus \{0\}\},$$
$$U_2 = X \setminus \{(0,z) \mid z \in \mathbb{C} \setminus \{0\}\},$$

Calculate $\check{H}^1(\mathfrak{U}, \mathcal{O}_X)$. Sketch an argument that $\check{H}^1(\mathfrak{U}, \mathcal{O}_X) \cong H^1(X, \mathcal{O}_X)$.

(c) Let X be a topological space, $x \in X$ a point and G a group. Consider the *skyscraper sheaf* $\mathcal{G}$ defined by

$$\mathcal{G}(U) = \begin{cases} G & x \in U, \\ 0 & x \notin U, \end{cases}$$

with restriction maps $\mathcal{G}(U) \to \mathcal{G}(V)$ being the identity whenever both groups are G. Verify that $\mathcal{G}$ satisfies the sheaf axioms. Calculate the stalks $\mathcal{G}_y$ of $\mathcal{G}$ for all points $y \in X$.

(d) Let X be a Riemann surface (i.e., $\dim_{\mathbb{C}} X = 1$). Denote by $\mathcal{M}_X$ the sheaf of meromorphic functions on X. As every holomorphic function is meromorphic, there is a natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{M}_X$. Describe the stalks of $\mathcal{M}_X/\mathcal{O}_X$, and describe this sheaf as a direct sum of skyscraper sheaves. Interpret the connecting homomorphism in the exact sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{M}_X) \to H^0(X, \mathcal{M}_X/\mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$$

in terms of the existence of meromorphic functions with certain specified properties.

5

Let $Y \subseteq X$ be a smooth hypersurface in a complex manifold X of dimension n, and let α be a meromorphic section of $K_X = \Omega^n_X$ the canonical line bundle of X. Assume α only has a pole along Y, and that this pole is simple (order one). Locally in a coordinate system $z_1, \ldots, z_n$ where Y is given by $z_1 = 0$ one can write

$$\alpha = h \cdot \frac{dz_1}{z_1} \wedge dz_2 \wedge \cdots \wedge dz_n$$

with $z_1 = 0$ defining Y and h a holomorphic function. We set

$$\mathrm{Res}_Y(\alpha) = (h \cdot dz_2 \wedge \cdots \wedge dz_n)|_Y.$$

(a) Show that $\mathrm{Res}_Y(\alpha)$ is well-defined and it yields an element of $\Gamma(Y, K_Y)$.

(b) Now let $X = \mathbb{P}^n$, and suppose Y is a smooth hypersurface defined by an irreducible homogeneous polynomial f of degree $n+1$. Show that

$$\alpha := \sum (-1)^i z_i f^{-1} dz_0 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n$$

can be interpreted as a meromorphic section of $K_{\mathbb{P}^n}$ with simple poles along Y. Furthermore, show that $\mathrm{Res}_Y(\alpha) \in H^0(Y, K_Y)$ is a nowhere vanishing section of K_Y. [*Hint: Make use of sections $Z : U \to \mathbb{C}^{n+1} \setminus \{0\}$ of the quotient map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. For the last statement, you may use without proof that if f is an irreducible homogeneous polynomial, then the equation $f = 0$ defines a smooth hypersurface Y if and only if $\partial f/\partial z_0, \ldots, \partial f/\partial z_n$ do not vanish simultaneously on Y.*]

END OF PAPER