UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 31 May, 2016 1:30 pm to 4:30 pm

PAPER 117

SPECTRAL GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let p be a point in a Riemannian manifold M. State how the value at p of the Laplacian of a function on M may be calculated using suitable geodesics through p.

Let S^d be the unit sphere in $\mathbb{R}^{d+1}$ with the induced metric. State and prove the relation between the action of the Laplacian $\tilde{\Delta}$ on functions in $\mathbb{R}^{d+1}$ and that of the Laplacian Δ on the restrictions of those functions to S^d.

Derive the spectrum of the Laplacian acting on smooth functions on S^d.

2

Define a flat d-dimensional torus. For such a torus state and prove what is its spectrum for the Laplacian acting on smooth functions.

Prove that isospectral flat two-dimensional tori are isometric.

3

State the transplantation theorem for the eigenfunctions of the Laplacian between manifolds constructed from copies of a Euclidean domain by identifying various pairs of boundary faces of those domains.

Describe the "propeller domains" in $\mathbb{R}^2$ and show that there is a 3-parameter family of pairs of such domains with the two members of each pair isospectral but not isometric.

4

A group T of order 168 has Gassman equivalent non-conjugate subgroups U_1 and U_2 of index 7. Given that T is generated by elements A and B where A, B and AB all have order 7, construct a pair of isospectral Riemann surfaces of genus 3.

Explain how your surfaces could be isometric and, if they are, how you could nevertheless obtain non-isometric surfaces of genus three.

5

Find the maximum number of pairwise disjoint simple closed geodesics that can occur in a Riemann surface of genus $g \geqslant 2$. Show that a maximal set of such geodesics determines a decomposition of S whose main features may be represented by a certain graph. Give two possible graphs for surfaces of genus three and indicate a diffeomorphism between the corresponding surfaces, explaining why this is not evident from the graphs.

Use this decomposition of S to identify parameters that determine S up to isometry. Define Teichmüller space, giving an important property of its analytic structure.

State Wolpert's theorem and identify two major steps in its proof, commenting on their role.

END OF PAPER