UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 2 June, 2016 1:30 pm to 4:30 pm

# PAPER 116

# MORSE THEORY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Suppose $f : M \to \mathbb{R}$ is a Morse function on a compact Riemannian manifold. For $x \in M$, let $\gamma_x(t)$ be the solution to the downward gradient flow equation $\gamma'(t) = -\nabla f|_{\gamma(t)}$ which satisfies $\gamma(0) = x$. Let $C_{[a,b]} = f^{-1}([a,b])$, and suppose that $|\nabla f| \geqslant \epsilon > 0$ on $C_{[a,b]}$. Show that there is some $T$ (depending only on $a, b$, and $\epsilon$) such that if $x \in C_{[a,b]}$, then $\gamma_x(t) \notin C_{[a,b]}$ for $t > T$.

Now let $U_\epsilon = \{x \in M \,|\, |\nabla f| < \epsilon\}$, and let $C_\epsilon = M \setminus U_\epsilon$ be its complement. Show that the total amount of time $\gamma_x(t)$ spends in $C_\epsilon$ is bounded independent of $x$.

Finally, suppose that $\lim_{t\to\infty} \gamma_x(t) = x_\infty$. Stating clearly any theorems you use, show that for an appropriate choice of Riemannian metric on $M$, $\gamma_x(t)$ decays exponentially to $x_\infty$; that is, there are constants $C$ and $k > 0$ such that $d(\gamma_x(t), x_\infty) < Ce^{-kt}$. Give an example to show that if $f$ is not Morse, $\gamma_x(t)$ need not decay exponentially to $x_\infty$.

**2**

Explain what it means to say that two handle decompositions of a manifold $M$ are related by a *handleslide.*

Suppose that we are given a handle decomposition of $M$ with $k$-handles $\mathcal{H}_{k,1}, \ldots \mathcal{H}_{k,r}$ and $k+1$-handles $\mathcal{H}_{k+1,1}, \ldots, \mathcal{H}_{k+1,s}$. Let $B_i$ be the belt sphere of $\mathcal{H}_{k,i}$, and let $A_j$ be the attaching sphere of $\mathcal{H}_{k+1,j}$. Suppose that $B_1$ intersects $A_1$ transversely in a single point. Without using the handle cancellation theorem, show that after a sequence of handleslides, we can obtain a new handle decomposition of $M$ with $k$-handles $\mathcal{H}'_{k,1}, \ldots \mathcal{H}'_{k,r}$ and $k+1$-handles $\mathcal{H}'_{k+1,1}, \ldots, \mathcal{H}'_{k+1,s}$ such that $A'_1$ intersects $B'_1$ transversely in one point, and $A'_j \cap B'_1 = \emptyset$ for $j > 1$.

By a further series of handleslides, show that we can obtain a new handle decomposition of $M$ with $k$-handles $\mathcal{H}''_{k,1}, \ldots \mathcal{H}''_{k,r}$ and $k+1$-handles $\mathcal{H}''_{k+1,1}, \ldots, \mathcal{H}''_{k+1,s}$ such that $A''_1$ intersects $B''_1$ transversely in one point, $A''_j \cap B''_1 = \emptyset$ for $j > 1$, and $A''_1 \cap B''_i = \emptyset$ for $i > 1$.

Express the intersection numbers $A''_l \cdot B''_k$ in terms of $A_j \cdot B_i$.

**3**

Suppose that $M$ is a Riemannian manifold of dimension $n$, and that $\gamma : [0,1] \to M$ is a geodesic. What is a *Jacobi field* on $\gamma$? Show that the space of Jacobi fields on $\gamma$ forms a vector space of dimension $2n$.

Stating clearly any theorems you use, show that if the image of $\gamma$ is contained in a geodesically convex open set $U$, then any Jacobi field on $\gamma$ is determined by its values at the endpoints of $\gamma$.

Now suppose that $M = SU(3)$ with its bi-invariant metric, and let $\gamma(t) = \exp(tA)$, where $A$ is a diagonal matrix with eigenvalues $2\pi i/3, 2\pi i/3$, and $-4\pi i/3$. Show that the space of Jacobi fields on $\gamma$ which vanish at both endpoints of $\gamma$ has dimension $\geqslant 4$.

**4**

Let $M$ be a Riemannian manifold, and let $\gamma : [0,1] \to M$ be a geodesic with $\gamma(0) = p, \gamma(1) = q$. Under what conditions is $\gamma$ a nondegenerate critical point of the energy functional $E : \Omega(p,q) \to \mathbb{R}$? Assuming that $\gamma$ is a nondegenerate critical point, explain clearly how to compute its index. (No proofs are needed, just the statements.)

Let $S^n$ be the sphere with the round metric, and let $p, q \in S^n$ with $p \neq -q$. Describe the set of critical points of $E$ in this case. What are their indices? Justify your answers. Hence or otherwise, compute $H_*(\Omega(S^n))$ for $n > 2$.

Stating clearly any results from homotopy theory that you use, show that $\pi_k(S^n) \simeq \pi_{k+1}(S^{n+1})$ for $k \leqslant 2n-2$.

**5**

Suppose that $L_0$ and $L_1$ are Lagrangian submanifolds of a symplectic manifold $(M,\omega)$, and let $\Omega(L_0, L_1)$ be the space of smooth paths from $L_0$ to $L_1$. Assuming that $\Omega(L_0, L_1)$ is path connected and that $H_2(M, L_0 \cup L_1) = 0$, define the *area functional* $A : \Omega(L_0, L_1) \to \mathbb{R}$. Under the given conditions, show that $A$ is well-defined up to the addition of a global constant. Determine (with proof) the set of critical points for the area functional.

What is meant by a *compatible almost complex structure* $J$ on $(M,\omega)$? Show that if $M$ has a Riemannian metric induced by a compatible almost complex structure $J$, then the set of formal solutions to the downward gradient flow equation for $A$ is given by $J$-holomorphic maps of the strip $\mathbb{R} \times [0,1] \to M$ with the property that $\mathbb{R} \times \{i\} \mapsto L_i$ for $i \in \{0,1\}$.

**END OF PAPER**