UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 30 May, 2016 1:30 pm to 4:30 pm

PAPER 115

DIFFERENTIAL GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

If M is a smooth manifold, explain what is meant by a *smooth vector field* on M. For $\alpha : M \to N$ a diffeomorphism and X a smooth vector field on M, define the vector field $\alpha_* X$ on N. Given two smooth vector fields X, Y on M, define the *Lie bracket* vector field $[X, Y]$ and state the corresponding *Jacobi identity.*

Given a smooth vector field X on M, describe without proof what is meant by a *local flow* $\phi_t : U \to \phi_t U$ determined by X (existence and uniqueness statements for such flows may be assumed). Define what is meant by a smooth *tensor* on M, and for any such tensor T define the *Lie derivative* $L_X T$. Identify (with proofs) $L_X T$ when $T = f$ is a smooth function and $T = Y$ is a smooth vector field.

If $\alpha : M \to N$ is a diffeomorphism and X a smooth vector field on M, with ϕ_t a corresponding local flow, show that the vector field $\alpha_* X$ has a corresponding local flow given by $\alpha \circ \phi_t \circ \alpha^{-1}$. If $M = N$, deduce that $\alpha_* X = X$ if and only if α and the local flows ϕ_t determined by X always commute.

2

Define the exterior derivative map d on p-forms on a smooth manifold M, showing that it is well-defined globally under your definition. Define what is meant by the de Rham cohomology groups $H^p_{DR}(M, \mathbf{R})$ of M. Assuming Green's theorem, show that if Δ is a 2-dimensional disc then $H^1_{DR}(\Delta, \mathbf{R}) = 0$. If M is compact of dimension n, state Stokes's theorem without boundary, showing that it defines a linear map $H^n_{DR}(M, \mathbf{R}) \to \mathbf{R}$.

Now let G denote a Lie group of dimension n. For $g \in G$, we let $L_g : G \to G$ be given by left-multiplication by g. Show that one can define a non-zero *invariant* n-form ω on G, that is a smooth n-form for which $(L_g)^* \omega = \omega$ for all $g \in G$ (you need not check that the n-form you define is actually smooth). Assume now the fact that for any n-form ω on G, there is an invariant n-form with the same de Rham cohomology class; when G is compact, deduce that $H^n_{DR}(G, \mathbf{R}) = \mathbf{R}$.

Using the results from this question, calculate de Rham cohomology groups $H^1_{DR}(S^2, \mathbf{R})$ and $H^3_{DR}(S^3, \mathbf{R})$.

3

Given a smooth curve $\gamma : [a,b] \to M$ on a smooth manifold, define what is meant by a *smooth vector field* $V(t)$ *along* γ. Given a Koszul connection ∇ on M, explain carefully the concepts of the *covariant derivative* of $V(t)$ along γ, and $V(t)$ being *parallel* along γ. Given a tangent vector $V_a \in T_{\gamma(a)}M$, show that there exists a unique parallel vector field $V(t)$ along γ with $V(a) = V_a$. Hence deduce the existence of parallel translation maps $\tau_t : T_{\gamma(a)}M \to T_{\gamma(t)}M$, which are isomorphisms of vector spaces.

Given two Koszul connections ∇ and $\widetilde{\nabla}$, show that there is a tensor Δ on M defined by $\Delta(X,Y) = \nabla_X Y - \widetilde{\nabla}_X Y$, where X and Y are local vector fields. A smooth curve γ is said to be a *geodesic* with respect to a Koszul connection ∇ if the smooth vector field $\dot{\gamma}(t)$ along γ is parallel. Explain briefly why is it true that a geodesic (together with parametrisation) is locally determined by its initial point and tangent vector. Show that the connections ∇ and $\widetilde{\nabla}$ have the same geodesics (with same parametrizations) if and only if $\Delta(v,v) = 0$ for all tangent vectors v.

4

Let M be an embedded smooth submanifold of a smooth manifold N. Explain how a Koszul connection ∇ on N induces a linear connection (which we shall also denote as ∇) on the bundle $TN|_M$, the restriction of TN to M. Suppose $\langle\ ,\ \rangle$ is a Riemannian metric on N, thus inducing a Riemannian metric on M and determining an orthogonal projection map $\pi : TN|_M \to TM$ of bundles on M. Show that $D = \pi \circ \nabla$ is a Koszul connection on M.

State the defining properties for the Levi–Civita connection on a Riemannian manifold. Suppose now in the above set-up, we take ∇ to be the Levi–Civita connection on N. For local vector fields V, W on M, we set $II(V,W)$ to be the normal component of $\nabla_V W$ with respect to the metric. Assuming the fact that D as defined above (namely where $D_V W$ is the tangential component of $\nabla_V W$) is the Levi–Civita connection on M, show that II induces a symmetric bilinear form on the tangent bundle of M with values in the normal bundle (called the *second fundamental form*). If R denotes the Riemannian curvature tensor on N and $\bar{R}$ the Riemannian curvature tensor on M, for tangent vectors v, w, x, y to M at P, prove that

$$\bar{R}(x,y,v,w) = R(x,y,v,w) + \langle II(v,x), II(w,y)\rangle - \langle II(v,y), II(w,x)\rangle.$$

[*You may use the fact that for a connection* ∇ *on a vector bundle* E*, with curvature operator denoted by* $\mathcal{R}$*, then for any section* σ *of* E*,*

$$\mathcal{R}(V,W)(\sigma) = \nabla_V \nabla_W \sigma - \nabla_W \nabla_V \sigma - \nabla_{[V,W]}\sigma. \quad]$$

END OF PAPER