UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 1 June, 2016 9:00 am to 12:00 pm

PAPER 114

ALGEBRAIC TOPOLOGY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $n \in \mathbb{N}$ be an integer with $n \geqslant 2$. Let $\hat{\alpha} : \mathbb{Z} \to \mathbb{Z}/n$ be the group homomorphism given by reduction modulo n, and let $\alpha : \mathbb{Z}/(n^2) \to \mathbb{Z}/n$ be the group homomorphism for which $\alpha(1) = 1$. Explain briefly why these yield long exact sequences

$$\cdots \to H^i(X;\mathbb{Z}) \to H^i(X;\mathbb{Z}) \xrightarrow{\hat{\alpha}} H^i(X;\mathbb{Z}/n) \xrightarrow{\hat{\beta}} H^{i+1}(X;\mathbb{Z}) \to \cdots$$

and

$$\cdots \to H^i(X;\mathbb{Z}/n) \to H^i(X;\mathbb{Z}/(n^2)) \xrightarrow{\alpha} H^i(X;\mathbb{Z}/n) \xrightarrow{\beta} H^{i+1}(X;\mathbb{Z}/n) \to \cdots .$$

for homomorphisms $\hat{\beta}$ and β as indicated.

(a) Show that for any $i \geqslant 1$ and n, there is a space X for which $\beta : H^i(X;\mathbb{Z}/n) \to H^{i+1}(X;\mathbb{Z}/n)$ is non-zero.

(b) By relating β and $\hat{\beta}$, show that the composite

$$H^i(X;\mathbb{Z}/n) \xrightarrow{\beta} H^{i+1}(X;\mathbb{Z}/n) \xrightarrow{\beta} H^{i+2}(X;\mathbb{Z}/n)$$

vanishes.

(c) Denote by $H\beta^*(X;n)$ the cohomology groups of the complex $\{H^*(X;\mathbb{Z}/n);\beta\}$. Compute $H\beta^*(\mathbb{RP}^3;2)$.

(d) Let M be a closed 3-dimensional manifold and suppose p is prime. What is the Euler characteristic of the graded vector space $H\beta^*(M;p)$? Justify your answer.

2

Let Σ_g denote a closed oriented surface of genus g.

(a) Compute $H^*(\Sigma_g; \mathbb{Z})$ as a ring, giving careful statements of any general theorems to which you appeal.

(b) Show that there is a degree one map $\Sigma_g \to \Sigma_h$ if and only if $g \geqslant h$.

(c) Let Σ_h^∂ denote the two-dimensional manifold-with-boundary obtained by removing an open disc from Σ_h. Let $\iota : \Sigma_h^\partial \subset \Sigma_g$ denote the inclusion of an embedded subsurface-with-boundary. Show that when $h > g/2$ there is no map $r : \Sigma_g \to \Sigma_h^\partial$ for which $r \circ \iota$ is equal to the identity of Σ_h^∂. Is this bound sharp? Justify your answer.

[*You may assume the following fact from linear algebra: if V is a real vector space equipped with a non-degenerate skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$, and $W \subset V$ is a subspace with $\langle u, v \rangle = 0$ for every $u, v \in W$, then* $\dim_{\mathbb{R}}(W) \leqslant \dim_{\mathbb{R}}(V)/2$.]

3

For each of the following assertions, provide a proof or a counterexample. General results from the course may be used without proof if clearly stated.

(a) The cohomology ring of complex projective space $H^*(\mathbb{CP}^n; \mathbb{Z})$ is $\mathbb{Z}[x]/(x^{n+1})$ for a generator x of degree 2.

(b) Let $m > n > 1$, let $\phi : S^m \to S^n$ and let X_ϕ be the cell complex obtained by attaching an $(m+1)$-cell to S^n via ϕ. The ring structure in $H^*(X_\phi; \mathbb{Z})$ is always independent of ϕ.

(c) Let U, V and W be complex vector spaces, and let $\phi : U \otimes V \to W$ be a $\mathbb{C}$-linear map. If for every non-zero $u \in U$ and non-zero $v \in V$ the restriction of ϕ to the subspaces $\{u\} \otimes V$ and $U \otimes \{v\}$ is injective, then

$$\dim_{\mathbb{C}}(\text{image}(\phi)) \geqslant \dim_{\mathbb{C}}(U) + \dim_{\mathbb{C}}(V) - 1.$$

4

Let M be an oriented smooth manifold. Explain how to associate a cohomology class $\varepsilon_Y \in H^*_{ct}(M;\mathbb{Q})$ to an oriented closed smooth submanifold $Y \subset M$.

Let M and N be closed connected oriented smooth n-dimensional manifolds, and $f, g : M \to N$ be smooth maps. Let $f^! : H^*(M;\mathbb{Q}) \to H^*(N;\mathbb{Q})$ denote $(D_N)^{-1} \circ f_* \circ D_M$, where $D_\bullet$ denotes the Poincaré duality isomorphism. Define $L(f,g)$ to be

$$L(f,g) \;=\; \sum_i (-1)^i \,\mathrm{Trace}\left(g^* f^! : H^i(M;\mathbb{Q}) \to H^i(M;\mathbb{Q})\right).$$

Prove that if $L(f,g) \neq 0$ then there is some $m \in M$ for which $f(m) = g(m)$.

Let $f : \mathbb{CP}^{2k} \to \mathbb{CP}^{2k}$ be a map of non-zero degree $d \neq 0$. By showing

$$L(f,f) = \chi(\mathbb{CP}^{2k}) \cdot d$$

where χ denotes the Euler characteristic, or otherwise, prove that if $g : \mathbb{CP}^{2k} \to \mathbb{CP}^{2k}$ is homotopic to f, the maps f and g co-incide at some point.

5

Define the Chern classes $c_i(E)$ of a complex vector bundle $E \to X$ over a space X. (You may assume that X admits a finite cover by open sets over which E is trivial. You should make sure that your answer explains why the Chern classes are well-defined.)

State without proof a result relating the Chern classes of the Whitney sum $E \oplus E'$ to the Chern classes of the bundles E and E'.

Equip $\mathbb{C}^n$ with its standard Hermitian inner product. Let $X \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ denote the space of pairs of orthogonal complex lines in $\mathbb{C}^n$. Prove that as a ring

$$H^*(X;\mathbb{Z}) \;=\; \mathbb{Z}[x,y]/I \qquad I = \left\langle x^n,\; x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}\right\rangle$$

where x, y have degree 2 and I is the given ideal.

END OF PAPER