UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 27 May, 2016 1:30 pm to 4:30 pm

PAPER 113

ALGEBRAIC GEOMETRY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

Throughout this paper, rings are commutative with element 1.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Let A be a ring. Give the definitions of $X = \text{Spec } A$, the *Zariski topology* on X, and the sheaf $\mathcal{O}_X$. Show that for any $p \in X$ the stalk $(\mathcal{O}_X)_p$ is isomorphic to A_p as rings (you do not need to show this is an isomorphism of *local* rings).

(ii) Let k be a field, let $\mathbb{A}^2_k = \text{Spec } k[t_1, t_2]$, and let X be the open subscheme $\mathbb{A}^2_k \setminus \langle t_1, t_2 \rangle$. Show that X is not an affine scheme.

(iii) Let $X = \text{Proj } \mathbb{R}[t_0, t_1, t_2]/\langle t_0^2 + t_1^2 + t_2^2 \rangle$. Show that X is not isomorphic to $\mathbb{P}^1_{\mathbb{R}}$.

2

(i) Give the definition of the *fibred product* of schemes, and give a sketch of its construction. Give an example of integral schemes X, Y, S and appropriate morphisms such that $X \times_S Y$ is not irreducible.

(ii) Let $f \colon X \to Y$ be a morphism of schemes, $y \in Y$, and X_y be the fibre of f over y. Show that X_y as a topological space is homeomorphic to $f^{-1}\{y\}$ with the induced topology as a subset of X.

3

(i) Let $S = \bigoplus_{d \geqslant 0} S_d$ be a graded ring which is generated by the elements of S_1 as an S_0-algebra, and let $X = \text{Proj } S$. Assume M and N are graded S-modules. Prove that $\mathcal{O}_X(n)$ is an invertible sheaf and that $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \simeq \widetilde{M \otimes_S N}$.

(ii) Let k be a field and let $X = \mathbb{P}^n_k$. Show that there is a Cartier divisor D on X such that $\mathcal{O}_X(D) \simeq \mathcal{O}_X(1)$.

(iii) Let X be a Noetherian scheme and let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of coherent sheaves on X. Give a proof or a counter-example (with justifications) for each of the following statements:

- If $\mathcal{F}$ and $\mathcal{G}$ are locally free, then $\mathcal{H}$ is locally free.
- If $\mathcal{F}$ and $\mathcal{H}$ are locally free, then $\mathcal{G}$ is locally free.

4 (i) Let $(X, \mathcal{O}_X)$ be a ringed space. Define what is meant by a *flasque sheaf* on X. Prove that if $\mathcal{I}$ is an $\mathcal{O}_X$-module which is injective in the category of $\mathcal{O}_X$-modules, then $\mathcal{I}$ is flasque.

(ii) Let X be a closed subset of a topological space Y and $f\colon X \to Y$ the inclusion map. Prove that for any sheaf $\mathcal{F}$ on X, we have $H^i(Y, f_*\mathcal{F}) \simeq H^i(X, \mathcal{F})$ for every i.

(iii) Define what is meant by a *skyscraper sheaf* on a topological space, and calculate its cohomology groups.

5

(i) Let k be a field, $f\colon X \to \mathbb{P}^n_k$ be a closed immersion, and $\mathcal{F}$ be a coherent sheaf on X. Moreover, let $\mathcal{O}_X(d) = f^*\mathcal{O}_{\mathbb{P}^n_k}(d)$ and put $\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$. Prove that $H^p(X, \mathcal{F}(d)) = 0$ for every $p > 0$ and $d \gg 0$. [You may assume $f_*(\mathcal{F}(d)) \simeq (f_*\mathcal{F})(d)$.]

(ii) Let k be a field and let $X = \mathbb{P}^n_k$ where $n > 0$. Find a coherent sheaf $\mathcal{F}$ on X such that

$$\dim_k H^0(X, \mathcal{F}) \neq \dim_k H^n(X, \mathcal{F}^\vee(-n-1))$$

where $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

(iii) Let X be the closed subscheme of $\mathbb{P}^3_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[t_0, \ldots, t_3]$ defined by the ideal $\langle t_2 - t_3, t_1^3 + t_0 t_2^2 \rangle$. Calculate the Hilbert polynomial $\Phi_{\mathcal{O}_X}$ of $\mathcal{O}_X$.

END OF PAPER