UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 31 May, 2016 1:30 pm to 3:30 pm

PAPER 112

PROBABILISTIC COMBINATORICS AND ITS APPLICATIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) State and prove the Prékopa–Leindler inequality.

(ii) Deduce the Brunn-Minkowski inequality (in any form).

(iii) Let $A, B \subset \mathbb{R}^n$ be non-empty open subsets. Is $\lambda(A + tB)$ an increasing function of t for $t > 0$?

2

For $n \geqslant 2$, let $B^n = \{x \in \mathbb{R}^n : \|x\| \leqslant 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the unit ball and unit sphere in $\mathbb{R}^n$ centred at the origin. Write $v_n = \mathrm{Vol}_n B^n$ for the 'volume' of B^n and $s_{n-1} = \mathrm{Vol}_{n-1} S^{n-1}$ for the 'surface area' of S^{n-1}, and recall that

$$s_{n-1} = n v_n \quad \text{and} \quad \sqrt{\frac{n}{2\pi}} < v_{n-1}/v_n < \sqrt{\frac{n+1}{2\pi}}.$$

For $0 < \rho < \pi/2$ write D_ρ for the spherical cap of spherical radius ρ on S^{n-1} centred at $(1, 0, \ldots, 0)$. Recall that

$$\mathrm{Vol}_{n-1}(D_\rho) \geqslant (\sin\rho)^{n-1} v_{n-1}.$$

[This can be seen by projecting D_ρ into the affine hyperplane $\{x \in \mathbb{R}^n : \ x_1 = 1\}$.]

(i) Show that for $\pi/4 \leqslant \rho < \pi/2$ we have

$$\mathrm{Vol}_{n-1}(D_\rho) \leqslant (\sin\rho)^n s_{n-1}. \tag{1}$$

(ii) Prove the stronger result that (1) holds if $\sin\rho \geqslant 1/\sqrt{n}$.

(iii) Let $r \geqslant \sqrt{2}$ and $P \subset \mathbb{R}^n$ a convex polyhedron such that

$$\frac{1}{r} B^n \subset P \subset B^n. \tag{2}$$

Show that P has at least $e^{n/(2r^2)}$ facets.

(iv) Show that there is a convex polyhedron in $\mathbb{R}^n$ with at most 3^n facets such that (2) is satisfied with $r = 2$.

3

i) State Talagrand's Inequality, introducing the concepts you need.

(ii) Let X_n be a set of n random points in the square $[0,1]^2$, and let $Y_n = \mathrm{TS}(X_n)$ be the length of a Travelling Salesman Tour through X_n. Prove that

$$Y_n \leqslant 2\sqrt{n} \quad \text{and} \quad \mathbb{E}(Y_n) \geqslant \sqrt{n}/5.$$

(iii) Show that for $t > 0$ we have

$$\mathbb{P}\big(|Y_n - M_{Y_n}| \geqslant t\big) \leqslant 4\ e^{-t^2/64},$$

where M_{Y_n} is the median of Y_n.

The results your proof needs should be stated precisely.

4

Let $k = k(n) = \lfloor \log\log n \rfloor$ and $p = p(n) = n^{-2/k}$. Show that whp the binomial random graph $G_{n,p}$ has clique number k, i.e., as $n \to \infty$,

$$\mathbb{P}\big(G_{n,p} \ \text{ contains a complete graph } K_k\big) = 1 + o(1), \tag{1}$$

and

$$\mathbb{P}\big(G_{n,p} \ \text{ contains a complete graph } K_{k+1}\big) = o(1). \tag{2}$$

END OF PAPER