UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2016 9:00 am to 11:00 am

PAPER 111

TECHNIQUES IN NON-ABELIAN ADDITIVE COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Prove that there exists a constant C such that for every $n \in \mathbb{N}$, if A is a subset of $\mathbb{F}_3^n$ of density at least Cn^{-1}, then A contains distinct elements x, y, z with $x+y+z=0$. [*You may assume the basic definitions and results of Fourier analysis on finite Abelian groups.*]

2 Let G be a finite bipartite graph, regular on each side, of density γ, with vertex sets X and Y. Prove that the following statements about G are equivalent, in the sense that if the ith statement holds with constant $c_i > 0$, then the jth statement holds for some constant $c_j > 0$ that is independent of $|X|$ and $|Y|$ and tends to zero with c_i.

(i) $|\mathbb{E}_{x,y}G(x,y)A(x)B(y) - \alpha\beta\gamma| \leqslant c_1$ for every pair of sets $A \subset X$ and $B \subset Y$ of densities α and β, respectively.

(ii) $\|G\|_{\square}^4 \leqslant \gamma^4 + c_2$.

(iii) The second largest singular value of the linear map θ_G, defined by the formula $(\theta_G f)(x) = \mathbb{E}_y G(x,y)f(y)$, is at most c_3.

3 Assuming any representation theory you might need, develop the basic theory of Fourier analysis for scalar-valued functions on finite groups. Use it to prove that if G has no non-trivial representation of dimension less than m, and A, B and C are subsets of G with $|A||B||C| > m^{-1}|G|^3$, then there exist $x \in A$, $y \in B$ and $z \in C$ with $xy = z$.

4 Let G be a finite group and let $f : G \to \mathrm{M}_n(\mathbb{C})$ be a function such that $\|f(x)\|_{\mathrm{op}} \leqslant 1$ for every $x \in G$, and

$$\mathbb{E}_{xy^{-1}zw^{-1}=e}\mathrm{tr}(f(x)f(y)^*f(z)f(w)^*) \geqslant cn.$$

Prove that there exist matrices $U(1), \ldots, U(r)$ and $V(1), \ldots, V(r)$ and irreducible unitary representations $\rho_1, \ldots, \rho_r$ with the following properties. (Write n_i for the dimension of the representation ρ_i.)

(i) For each p, $U(p)$ and $V(p)$ are $n \times n_p$ matrices.

(ii) $m = \sum_{p=1}^{r} n_p \in [cn/2, 2n/c]$.

(iii) For each p, $\mathbb{E}_x f(x)U(p)\rho_p(x)^* = \lambda_p V(p)$ for some real number λ_p that lies between $(c/2)^{1/2}$ and 1.

(iv) $\langle U(p), U(q)\rangle = \langle V(p), V(q)\rangle = n_p\delta_{pq}$ for each p and q with $\rho_p = \rho_q$.

Let U be the $n \times m$ matrix $(U(1)|\ldots|U(r))$ and for each x let $P(x)$ be the block-diagonal matrix $\rho_1(x) \oplus \cdots \oplus \rho_r(x)$. Prove that

$$\mathbb{E}_x\|UP(x)^*b\|_2^2 = \|b\|_2^2$$

for every $b \in \mathbb{C}^m$. [*You may assume any results from representation theory that you might need, provided that you state them clearly.*]

END OF PAPER