UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS    Part III

Friday, 27 May, 2016   9:00 am to 11:00 am

## PAPER 110

## EXTREMAL GRAPH THEORY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

State the Erdős-Stone theorem.

Let $t, r \in \mathbb{N}$ be given and let $(G_n)$ be a sequence of graphs, such that $G_n$ has order $n$ and size $(1-1/r+o(1))\binom{n}{2}$ Show that if $K_{r+1}(t)$ is not a subgraph of $G_n$ then $G_n$ contains an $r$-partite subgraph of minimum degree $(1-1/r+o(1))n$.

Let $F$ be a fixed graph with $\chi(F) = r+1$. Let $(H_n)$ be a sequence of graphs, such that $H_n$ has order $n$, $H_n$ does not contain $F$, and $e(H_n) = \mathrm{ex}(n, F)$. Show that $\delta(H_n) = (1-1/r+o(1))n$.

Show further that, if $F$ has a vertex $v$ such that $\chi(F-v) = r$, then $\Delta(H_n) = (1-1/r+o(1))n$.

**2**

Define an $\epsilon$-uniform pair.

Let $\lambda > 0$ and $r \in \mathbb{N}$. Let $G$ be a graph containing disjoint vertex sets $V_1, \ldots, V_r$, such that $|V_i| = n$, $1 \leqslant i \leqslant r$. Show that there exists $\eta > 0$ and $\delta > 0$ such that, if $(V_i, V_j)$ is $\eta$-uniform and $d(V_i, V_j) \geqslant \lambda$ for $1 \leqslant i < j \leqslant r$, and if $n$ is large, then $G$ contains at least $\delta n^r$ copies of $K_r$.

State Szemerédi's Regularity Lemma.

Deduce that, given $\epsilon > 0$, there exists $\delta > 0$ such that, for $n$ sufficiently large, if $H$ is a graph of order $n$ containing fewer than $\delta n^r$ copies of $K_r$, then there is some set of at most $\epsilon n^2$ edges of $H$, whose removal leaves a graph containing no $K_r$.

**3**

Let $t, N \in \mathbb{N}$, $t \geqslant 3$ and let $r = \binom{t}{2}$. Let $G(N, K_t)$ be the $r$-uniform hypergraph with vertex set $V = [N]^{(2)}$ and edge set $E = \{[T]^{(2)} : T \in [N]^{(t)}\}$. (Here we use the customary notation $[n] = \{1, \dots, n\}$ and $[n]^{(k)} = \{X \subset [n] : |X| = k\}$.)

Show that each vertex of $G(N, K_t)$ has degree $d = \binom{N-2}{t-2}$.

Show moreover that, given a subset $\sigma \subset V$, where $2 \leqslant |\sigma| \leqslant r$, then either $d(\sigma) = 0$ or $d(\sigma) = \binom{N-v}{t-v}$, where $v$ is some number depending on $\sigma$ that satisfies $3 \leqslant v \leqslant t$ and $|\sigma| \leqslant \binom{v}{2}$.

Derive the inequality $2(|\sigma|-1) \leqslant (v-2)(t+1)$. Deduce that, for any constant $c$, there is a constant $c'$ such that, if $\tau = c' N^{-2/(t+1)}$ and $N$ is large enough, then $d(\sigma) \leqslant c d \tau^{|\sigma|-1}$ holds for all $\sigma$.

What is meant by a set of containers for $G(N, K_t)$? State a theorem about the existence of such a set, in which each container has fewer than $\epsilon e(G(N, K_t))$ edges.

Deduce that the number of $K_t$-free graphs on vertex set $[N]$ is $2^{(1-1/(t-1)+o(1))\binom{N}{2}}$. [If you use any additional results in your argument, you need not prove them, but state them clearly.]

**4**

Let $G$ be an $r$-uniform hypergraph with vertex set $[n]$ and average degree $d$. Define the degree measure $\mu(S)$ of a subset $S \subset V(G)$. Prove that

$$\left(\mu(S) - 1 + \frac{1}{r}\right) nd \leqslant e(G[S]) \leqslant \mu(S)\frac{nd}{r}.$$

State a theorem giving a sufficient condition for the existence of containers for $G$ of measure at most $1 - c$.

State the golden rule paradigm, and explain how it can be used to produce containers. State an explicit algorithm which produces containers satisfying the theorem. Your algorithm should involve $s$-uniform multigraphs with edge multisets $P_s$, as well as parameters $\tau$ and $\delta$ such that $d(\sigma) \leqslant \delta d \tau^{|\sigma|-1}$ for $\sigma \subset [n]$, $|\sigma| \geqslant 2$.

Writing $d_s(u)$ for the degree in $P_s$ of a vertex $u$, show that

$$\sum_{u \in U} d_s(u) \leqslant \tau^{r-s} nd(\mu(U) + r\delta)$$

holds for all $U \subset [n]$.

**5**

Let $G(n, \ell, r)$ be the $\ell$-uniform hypergraph with $n$ vertices, whose vertex set is split into $r$ classes $V_1, \ldots, V_r$ with $|V_1| \leqslant |V_2| \leqslant \cdots \leqslant |V_r| \leqslant |V_1| + 1$, and whose edges are those $\ell$-sets $e$ for which there is an index $j$ such that $\sum_{i=1}^{k} |e \cap V_{j+i}| \geqslant k - 1$, for $1 \leqslant k \leqslant \ell - 1$. (Subscripts are taken modulo $r$.)

Prove that every set of $r+1$ vertices of $G(n, \ell, r)$ contains an edge.

Prove that $(G(n, \ell, r)$ has $(1 + o(1)) \left(\frac{\ell-1}{r}\right)^{\ell-1} \binom{n}{\ell}$ edges.

Deduce that the limiting extremal density $\pi(K_r^\ell)$ of $K_r^\ell$ satisfies $\pi(K_r^\ell) \geqslant 1 - \left(\frac{\ell-1}{r-1}\right)^{\ell-1}$. State a corresponding upper bound and explain, without performing detailed calculations, how it is proved.

## END OF PAPER