UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 30 May, 2016 1:30 pm to 3:30 pm

PAPER 109

COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the Local LYM inequality, and deduce the LYM inequality. Which antichains in $\mathcal{P}([n])$ have size exactly $\binom{n}{\lfloor n/2 \rfloor}$?

Let $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ be subsets of $[n]$ such that A_i meets B_j if and only if $i \neq j$. Prove that $\sum_i \binom{|A_i|+|B_i|}{|A_i|}^{-1} \leqslant 1$.

[*Hint: For each* i*, consider the permutations of* $[n]$ *for which all elements of* A_i *come before all elements of* B_i*.*]

Use this result to give another proof of the LYM inequality.

2

State the vertex-isoperimetric inequality in the discrete cube (Harper's theorem). Explain carefully how the Kruskal-Katona theorem may be deduced from Harper's theorem.

State the Erdős-Ko-Rado theorem, and give a proof of it using the Kruskal-Katona theorem.

A family $A \subset [k]^n$ is called *intersecting* if for any $x, y \in A$ we have $x_i = y_i$ for some i. How large can an intersecting family in $[k]^n$ be?

A family $A \subset [k]^n$ is called *weakly intersecting* if for any $x, y \in A$ we have $x_i, y_i > 1$ for some i. Show that, for n odd, the largest weakly intersecting family in $[k]^n$ is the family consisting of all x with $x_i > 1$ for more than $n/2$ values of i.

[*Hint: For each* $x \in A$*, consider* $\{i \in [n] : \ x_i > 1\}$*.*]

3

State and prove the edge-isoperimetric inequality in the discrete cube. Deduce that, among families $A \subset Q_n$ with $|A|$ given, the number of edges contained in A is maximised when A is an initial segment of the binary ordering.

[Here as usual we say that an edge xy is *contained* in A if $x, y \in A$.]

A *face* of Q_n is a 4-cycle (equivalently, it consists of four points of the form $x, x \cup \{i\}, x \cup \{j\}, x \cup \{i, j\}$, where i, j are distinct elements not belonging to x). Prove that, among families $A \subset Q_n$ with $|A|$ given, the number of faces contained in A is maximised when A is an initial segment of the binary ordering.

[*Hint: Use compressions as in the proof of the edge-isoperimetric inequality.*]

4

State and prove the Frankl-Wilson theorem (on modular intersections).

Let $A, B \subset \mathcal{P}([n])$ be families of sets such that $|x \cap y|$ is even for every $x \in A$ and $y \in B$. By considering the characteristic vectors (indicator functions) of the points of A and B, prove that $|A||B| \leqslant 2^n$.

Show that if instead we have $|x \cap y|$ odd for every $x \in A$ and $y \in B$ then

$$|A||B| \leqslant 2^{n+1}.$$

END OF PAPER