UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2016 1:30 pm to 4:30 pm

PAPER 108

TOPICS IN ERGODIC THEORY

*Attempt **ALL** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the *maximal ergodic theorem.*

State the *pointwise ergodic theorem.*

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving system. Let $f : X \to \mathbb{R}_{\geqslant 0}$ be a measurable function. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int f d\mu.$$

for μ-almost all x. Note that $\int f d\mu$ may be infinite.

[*You may use without proof the L^1 version of the mean ergodic theorem, as well as the pointwise ergodic theorem.*]

2

State *Szemerédi's theorem.*

State *Furstenberg's multiple recurrence theorem*, and prove that it implies Szemerédi's theorem.

[*If you construct a measure preserving system in your proof, you may omit the proof of the measure preserving property.*]

Prove the multiple recurrence theorem for the circle rotation.

3

Define *unique ergodicity.*

Prove that irrational circle rotations are uniquely ergodic.

Let $\alpha \in (0, 1)$ be an irrational number. Consider the map $T : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ defined by

$$T(x, y) = \{x + \alpha \mod 1, x + y \mod 1\}.$$

Prove that the measure preserving system $(\mathbb{R}^2/\mathbb{Z}^2, \mathcal{B}, m_2, T)$ is ergodic, where $\mathcal{B}$ is the Borel σ-algebra and m_2 is the Lebesgue measure.

[*You do not need to prove that m_2 is T-invariant.*]

Prove that the system $(\mathbb{R}^2/\mathbb{Z}^2, T)$ is uniquely ergodic.

[*You may NOT use without proof Furstenberg's theorem on unique ergodicity of skew products, but you may use without proof the basic properties of generic points.]*

END OF PAPER