UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2016 1:30 pm to 4:30 pm

PAPER 107

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the Hopf boundary point lemma for $u : \Omega \cup \{y\} \to \mathbb{R}$ with $u \in C^2(\Omega) \cap C^0(\Omega \cup \{y\})$ solving $Lu = a^{ij} D_{ij} u + b^i D_i u \geqslant 0$ in Ω. Be sure to state any extra hypothesis needed.

[You may use the comparison principle concerning functions f, g satisfying $Lf \geqslant Lg$ (as proven in class) without proof.]

2

Let $\Omega \subset \mathbb{R}^2$ be an open domain satisfying an exterior cone condition, i.e. for every $x_0 \in \partial\Omega$ there exists a solid cone C in $\mathbb{R}^2$ with vertex at x_0 such that $\overline{C} \cap \overline{\Omega} = \{x_0\}$. Consider the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}, \tag{1}$$

where f is a bounded function of class $C^{0,\mu}(\overline{\Omega})$, for a given $\mu \in (0,1)$, and $\varphi \in C^0(\partial\Omega)$.

(i) Show that at each $x_0 \in \partial\Omega$ there exists a barrier determined by a function of the form $g(r,\theta) = r^\nu h(\theta)$, where $r = |x - x_0|$ and θ is the angle between the vector $x - x_0$ and the axis of the exterior cone. In other words find g of the above form satisfying the following conditions: $g < 0$ in Ω, $g = 0$ at x_0 and $\Delta g \geqslant \delta > 0$ in Ω, for a positive δ (allowed to depend on Ω).

Hint: Set polar coordinates centred at x_0 with $\theta = 0$ corresponding to the axis of the cone and let $0 < \alpha < \pi$ be the opening angle of the cone, i.e. $\overline{C} = \{(r,\theta) \in [0,\infty) \times [-\pi,\pi] : -\alpha \leqslant \theta \leqslant \alpha\}$. You may use without proof the fact that the Laplacian in polar coordinates of a function $g(r,\theta)$ has the expression $\Delta g = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}$.

(ii) Give the defining expression of the Perron solution u to the Dirichlet problem (1) (you are not required to prove that $u \in C^2$ or u solves $\Delta u = f$).

(iii) Prove that u is continuous in $\overline{\Omega}$ and it satisfies the boundary condition $u|_{\partial\Omega} = \varphi$.

Hint: employ the barrier function constructed in part (i). You may use without proof the fact that, given a subfunction u_1 and a superfunction u_2, we have $u_1 \leqslant u_2$.

(iv) Show that the Dirichlet problem (1) admits a unique solution in $C^2(\Omega) \cap C^0(\overline{\Omega})$ (you may use any result proven throughout the course).

3

Let $n \geqslant 2$. Recall that

$$B_r^+ := \{x \in \mathbb{R}^n : |x| < r, x_n \geqslant 0\}$$
$$S_r := \{x \in \mathbb{R}^n, |x| < r, x_n = 0\}.$$

(a) Suppose that $u \in C^3(B_2^+)$ satisfies

$$\begin{cases} \Delta u = f & \text{in } B_2^+ \\ u = 0 & \text{on } S_2 \end{cases}$$

for $f \in C^1(B_2^+)$. Prove that

$$\|u\|_{W^{2,2}(B_1^+)} \leqslant C\left(\|u\|_{W^{1,2}(B_2^+)} + \|f\|_{L^2(B_2^+)}\right)$$

for some constant $C = C(n) \in (0, \infty)$. *Hint: First establish*

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n}\int_{B_1^+}(D_{ij}u)^2 \leqslant C\left(\|u\|^2_{W^{1,2}(B_2^+)} + \|f\|^2_{L^2(B_2^+)}\right).$$

(b) Suppose that $u \in C^3(B_2^+)$ satisfies

$$\begin{cases} \Delta u = f & \text{in } B_2^+ \\ u = \varphi & \text{on } S_2 \end{cases}$$

for $f \in C^1(B_2^+)$ and $\varphi \in C^3(B_2^+)$. Prove that

$$\|u\|_{W^{2,2}(B_1^+)} \leqslant C\left(\|u\|_{W^{1,2}(B_2^+)} + \|f\|_{L^2(B_2^+)} + \|\varphi\|_{W^{2,2}(B_2^+)}\right)$$

for some constant $C = C(n) \in (0, \infty)$.

4

Let $u : \Omega \to \mathbb{R}$ be a C^1 function on the open set $\Omega \subset \mathbb{R}^n$ that satisfies the weak form of the minimal surface equation, i.e.

$$\sum_{i=1}^{n} \int_{\Omega} \frac{D_i u}{\sqrt{1+|Du|^2}} D_i \zeta = 0 \quad \text{for any } \zeta \in C^1_c(\Omega).$$

Let $\ell \in \{1, ..., n\}$ and denote with $\{e_1, ..., e_n\}$ the standard orthonormal basis for $\mathbb{R}^n$. Consider difference quotients of u, i.e. for $l \in \{1, ...n\}$ and $h > 0$ define on $\Omega'' = \{y \in \Omega : \text{dist}(y, \partial\Omega) > 2h\}$ the difference quotient in the direction l:

$$\delta_{l,h} u(x) := \frac{u(x + he_\ell) - u(x)}{h}.$$

(i) Prove that $\delta_{l,-h} u$ solves a uniformly elliptic PDE in divergence form on Ω''.

Hint: use the test function $(\delta_{\ell,h}\zeta)(x) := \frac{\zeta(x+he_\ell)-\zeta(x)}{h}$ for $h > 0$ and $\zeta \in C^1_c(\Omega'')$, where $\Omega'' = \{y \in \Omega : \text{dist}(y, \partial\Omega) > 2h\}$. You may use the "discrete integration by parts formula"

$$-\int_{\Omega} g\,(\delta_{\ell,h} f) = \int_{\Omega} (\delta_{\ell,-h} g)\, f,$$

that holds for $f \in C^0_c(\Omega')$, where $\Omega' = \{y \in \Omega : \text{dist}(y, \partial\Omega) > h\}$ and $g \in C^0(\Omega)$.

(ii) Explain in a few sentences how you would deduce, from the conclusion obtained in part (i), that u is actually $C^{1,\alpha}$ in Ω. You are not required to fill in the details.

5

In this question you are free to use results proven in class without proof.

(a) State precisely the *Harnack inequality* relating $\sup_{B_1(0)} u$ to $\inf_{B_1(0)} u$ for u a weak solution on $B_2(0)$ to a PDE of the form $D_i(a^{ij} D_j u) = 0$. Be sure to state all extra hypotheses needed.

Suppose, for $n \geqslant 2$, that $\Omega \subset \mathbb{R}^n$ is a smooth, bounded, connected domain. Suppose that $u \in C^2(\overline{\Omega})$ satisfies $u \geqslant 0$ and solves

$$\begin{cases} Lu := a^{ij} D_{ij} u + b^i D_i u + cu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for $a^{ij}, b^i, c, f \in C^{0,\mu}(\overline{\Omega})$ with $\lambda|\xi|^2 \leqslant a^{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2$ for $\lambda, \Lambda \in (0, \infty)$.

(b) If $f \leqslant 0$ and u is not identically zero, show that $u > 0$.

(c) For f of arbitrary sign and $\Omega' \Subset \Omega$, use an argument by contradiction to show that there is $C = C(n, L, \Omega', \Omega)$ such that

$$\sup_{\Omega} u \leqslant C \left(\inf_{\Omega'} u + |f|_{0,\mu;\Omega} \right).$$

Note that the PDE under consideration is not of the "divergence form" considered in (a).

6

Let $n \geqslant 3$ and consider Ω a bounded domain with smooth boundary. Recall that the Sobolev inequality says that for $f \in W_0^{1,2}(\Omega)$,

$$\left(\int_\Omega |f|^{2\kappa}\right)^{\frac{1}{\kappa}} \leqslant C \int_\Omega |Df|^2$$

for some $C = C(n) \in (0, \infty)$ and $\kappa = \frac{n}{n-2}$.

Consider a Dirichlet eigenfunction for Ω, i.e., $u \in C^\infty(\overline{\Omega})$ solving

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume that u does not identically vanish.

(i) Show that $\lambda > 0$.

(ii) Prove that for $\gamma \geqslant 2$,

$$\left(\int_\Omega |u|^{\gamma\kappa}\right)^{\frac{1}{\kappa}} \leqslant C\lambda \frac{\gamma^2}{(\gamma - 1)} \int_\Omega |u|^\gamma$$

for some $C = C(n) \in (0, \infty)$.

(iii) Iterate this inequality to prove that

$$\sup_\Omega |u| \leqslant C\lambda^{\frac{n}{4}} \|u\|_{L^2(\Omega)}$$

for some $C = C(n) \in (0, \infty)$. *Hint:*

$$\prod_{j=0}^{\infty} (2\kappa^j)^{2^{-1}\kappa^{-j}} = (2\kappa)^{2^{-1}\sum_{j=0}^{\infty}\frac{j}{\kappa^j}} < \infty.$$

END OF PAPER