UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 30 May, 2016 9:00 am to 12:00 pm

PAPER 106

FUNCTIONAL ANALYSIS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let A be a unital Banach algebra. Prove that the group $G(A)$ of invertible elements of A is open and that the map $x \mapsto x^{-1} \colon G(A) \to G(A)$ is continuous.

Let A be a Banach algebra and $x \in A$. Define the *spectrum* $\sigma_A(x)$ *of* x *in* A both in the case that A is unital and in the case A is not unital.

Let A be a Banach algebra and $x \in A$. Prove that $\sigma_A(x)$ is a non-empty compact subset of $\mathbb{C}$. [You may assume any version of the Hahn–Banach theorem without proof.]

State and prove the Gelfand–Mazur theorem concerning complex unital normed division algebras.

Let K be an arbitrary set. Let A be an algebra of complex-valued functions on K with pointwise operations and assume that $\|\cdot\|$ is a complete algebra norm on A. Prove carefully that $\sup_K |f| \leqslant \|f\|$ for all $f \in A$.

2

State and prove Mazur's theorem on the weak-closure and norm-closure of a convex set in a Banach space. [Any version of the Hahn–Banach theorem can be assumed without proof provided it is clearly stated.]

Let (x_n) be a sequence in a Banach space X. A sequence (u_n) is called a *convex block of* (x_n) if there exist $p_1 < q_1 < p_2 < q_2 < \dots$ in $\mathbb{N}$ and non-negative real numbers $a_1, a_2, \dots$ such that $\sum_{i=p_n}^{q_n} a_i = 1$ and $\sum_{i=p_n}^{q_n} a_i x_i = u_n$ for all $n \in \mathbb{N}$. Prove that if $x_n \xrightarrow{w} 0$ then there is a convex block (u_n) of (x_n) such that $u_n \to 0$ in norm.

Let X be a Banach space whose dual space X^* is separable, and let (x_n) be a bounded sequence in X. Explain briefly by quoting appropriate theorems why there is a subsequence (y_n) of (x_n) that w^*-converges to some element ϕ in X^{**}. Show that if (u_n) is a convex block of (y_n), then $y_n - u_n \xrightarrow{w} 0$.

Let X be a Banach space with separable dual space X^*. Let Y be a closed subspace of X, let Z denote the quotient space X/Y and let $q \colon X \to Z$ be the quotient map. Let (z_n) be a sequence in the closed unit ball B_Z of Z such that $z_n \xrightarrow{w} 0$. Prove that there is a sequence (x_n) in $3B_X$ and a subsequence (z_{k_n}) of (z_n) such that $x_n \xrightarrow{w} 0$ and $\|q(x_n) - z_{k_n}\| \to 0$.

3

State the Radon–Nikodym theorem explaining the terms used.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that the dual space L_p^* of $L_p = L_p(\Omega, \mathcal{F}, \mu)$ is isometrically isomorphic to L_q.

Let f and $f_1, f_2, \ldots$ be members of L_p. Assume that $|f_n| \leqslant |f|$ for every n and that $f_n \to 0$ pointwise almost everywhere. Show that $f_n \xrightarrow{w} 0$.

4

Define the terms *hermitian, unitary* and *normal* describing an element x in a unital C^*-algebra.

Let A be a unital C^*-algebra, and let $x \in A$. Show that if x is hermitian, then $\sigma_A(x) \subset \mathbb{R}$, and if x is unitary, then $\sigma_A(x) \subset \mathbb{T}$. Deduce that if B is a unital C^*-subalgebra of A, and $x \in B$ is normal, then $\sigma_B(x) = \sigma_A(x)$.

Let T be a normal operator on a Hilbert space H. Let $K = \sigma(T)$ be the spectrum of T. Prove that there is a unique unital $*$-homomorphism $f \mapsto f(T) \colon C(K) \to \mathcal{B}(H)$ such that $u(T) = T$, where $u(z) = z$ for all $z \in K$.

Show that if K is disconnected then T has a non-trivial invariant subspace: there is a closed subspace Y of H such that $Y \neq \{0\}$, $Y \neq H$, and $Tx \in Y$ for all $x \in Y$.

[Throughout this question you may use without proof any result from the spectral threory of general Banach algebras including the Gelfand representation theorem, but you may not assume without proof any result specific to C^*-algebras.]

5

Let K be a weakly compact subset of a Banach space X. Explain briefly why K is w^*-closed when viewed as a subset of X^{**}.

Let $T\colon X \to Y$ be a bounded linear map between Banach spaces. Prove that the dual map $T^*\colon Y^* \to X^*$ is w^*-to-w^*-continuous.

State Goldstine's theorem and the Banach–Alaoglu theorem.

Let $T\colon X \to Y$ be a bounded linear map between Banach spaces. We say that T is *weakly compact* if $\overline{TB_X}$ is a weakly compact subset of Y. Show that the following are equivalent.

(i) T is weakly compact.

(ii) $T^{**}(X^{**}) \subset Y$.

(iii) $T^*\colon Y^* \to X^*$ is w^*-to-w-continuous.

(iv) T^* is weakly compact.

[Hint: for (i) $\Longrightarrow$ (ii) start with the observation that $B_X \subset (T^{**})^{-1}\big(\overline{TB_X}\big)$*.]*

Show also that if X or Y is reflexive then every bounded linear map $X \to Y$ is weakly compact.

END OF PAPER