UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 26 May, 2016 1:30 pm to 4:30 pm

PAPER 105

ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

*There are **THREE** questions in total.*
Question 2 is divided into 2a, 2b and 2c.
Question 3 is divided into 3a and 3b.

Attempt 1, 2a, 2b and 3a
*and **EITHER** 2c **OR** 3b.*

Question 1 is worth 40 marks and
questions 2 and 3 are worth 50 marks each.
2c and 3b are both worth 16 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State what it means for a hypersurface to be *non-characteristic* for a given constant coefficient linear partial differential operator. Which of the following Cauchy problems is non-characteristic:

(i) $u_t = u_{xxx}\,, u(x,0) = g(x)\,,$ where $u(x,t) \in \mathbb{R}$;

(ii) $i\psi_t + \psi_{xx} = 0\,, \psi(x,0) = g(x)\,,$ where $\psi(x,t) \in \mathbb{C}$;

(iii) $\phi_t - i\phi_x = 0\,, \phi(x,0) = g(x)\,,$ where $\phi(x,t) \in \mathbb{C}$.

(In all of these $(x,t) \in \mathbb{R}^2$.)

Show that for any solution $\phi \in C^1(\mathbb{R}^2;\mathbb{C})$ of (iii), the initial value $g = g(x)$ is necessarily real analytic, i.e. given by a convergent power series in a neighbourhood of every point. [*Hint: consider ϕ as a function of the complex variable $z = x + it$.*]

Write down a family $\{\phi_n(x,t)\}_{n=1}^{\infty}$ of C^1 solutions of (iii) such that, if $g_n(x) = \phi_n(x,0)$, then

$$\lim_{n\to\infty}\Big(\sum_{j=0}^{r}\sup_{x\in\mathbb{R}}|\partial_x^j g_n(x)|\Big) = 0\,, \forall r \in \mathbb{N}$$

but for each $t \neq 0$ there holds $\lim\limits_{n\to\infty} \sup_{x\in\mathbb{R}} |\phi_n(x,t)| = +\infty\,.$

State and prove the Cauchy estimates for the derivatives $f^{(j)}(z_0)$ of a function f at the centre of a disc $\overline{D}_r(z_0) = \{z : |z - z_0| \leqslant r\} \subset \mathbb{C}$ when f is holomorphic in an open set containing the disc and is bounded by M, on the disc i.e., $\sup_{\overline{D}_r(z_0)} |f(z)| \leqslant M$.

Now let $x \in \mathbb{C}$ and $t \in \mathbb{C}$ be complex variables, and consider the problem of proving existence of holomorphic solutions $\phi(x,t) \in \mathbb{C}$ of the equation

$$\phi_t - i\phi_x = f(x,t) \tag{1}$$

by studying the fixed point problem $\phi = T[\phi]$, where T is the integral operator

$$T[\phi](x,t) \;=\; \int_0^t \Big(i\phi_x(x,z) + f(x,z) \Big) dz\,.$$

Here we assume that $f(x,t)$ is holomorphic in an open set which contains $\{(x,t) : |x| \leqslant R, |t| \leqslant \eta\}$, so that the integral defining T in the complex plane is actually independent of the path chosen in this set. Deduce from the Cauchy estimate that, for a continuous choice of $\sigma(\tau) \in (0,1)$ satisfying $\sigma(\tau) > s > 0$, there holds

$$\sup_{|x|<sR} |(T[\phi] - T[\psi])(x,t)| \;\leqslant\; \frac{1}{R}\int_0^{|t|}\Big(\frac{1}{\sigma(\tau) - s}\Big) \times \Big(\sup_{\substack{|x|\leqslant\sigma(\tau)R \\ |z|\leqslant|\tau|}} |(\phi - \psi)(x,z)| \Big) d\tau\,,$$

The integral on the right is now a real integral, and it is understood that $|t| < \eta$ and ϕ, ψ are any pair of functions which are holomorphic for $|t| < \eta$ and $|x| < R$. Explain how this allows a proof that the operator T is a contraction in the Banach space $\mathcal{B}_\alpha$ of holomorphic functions on the domain

$$\mathcal{C}_\alpha = \{(x,t) \in \mathbb{C} \times \mathbb{C} : |t| < \alpha(1 - |x|/R) \text{ and } |x| < R\},$$

with norm

$$\|u\|_\alpha \stackrel{\text{def}}{=} \sup_{0<s<1} \sup_{|t|<\alpha(1-s)} \sup_{|x|\leqslant sR} \left[|u(x,t)| \frac{\alpha(1-s) - |t|}{|t|} \right],$$

for an appropriate choice of α. Deduce the Cauchy-Kovalevskaya theorem for (1), i.e. the existence of a local holomorphic solution if f is holomorphic, (for the case of complex x, t.)

Now consider (iii), with x and t again *real variables* and g real analytic, and deduce the existence of a local real analytic solution $\phi(x,t)$.

2

(a) Define the Sobolev space $W^{s,p}(\mathbb{R}^d)\,, 1 \leqslant p \leqslant \infty\,, s \in \mathbb{N}\,.$

State the Morrey inequality and the Sobolev inequality for a function $u \in W^{1,p}(\mathbb{R}^d)$, giving the conditions on p and d for them to hold. Prove the Morrey inequality.

Prove that if $u \in W^{1,p}(\mathbb{R}^d)$ for $p > d$ then $\lim_{|x|\to\infty} |u(x)| = 0$ (possibly after redefinition on a null set.)[*Hint: you may find it useful to prove first that if* $|u(x_0)| = 2\theta$ *for some positive number* θ*, then there exists a positive number* ρ *which is independent of* x_0 *such that* $|u(x)| \geqslant \theta$ *for* $|x - x_0| \leqslant \rho$*, and then argue by contradiction.*]

(b) State the Arzelà-Ascoli theorem, and assuming its validity:

(i) state the Rellich-Kondrachov compactness theorem and explain very briefly the main points in its proof;

(ii) prove that if $\{u_n : \mathbb{R}^d \to \mathbb{R}\}_{n=1}^{\infty}$ is a sequence of functions with $\sup_n \|u_n\|_{W^{1,p}(\mathbb{R}^d)} = M < \infty$, with $p > d$, then there is a subsequence $u_{n(j)}$ which converges uniformly on the unit ball $\{x : |x| \leqslant 1\}$ to a continuous limit (possibly after redefinition on a null set.)

Show that if u_n is a sequence in a Hilbert space X which converges weakly to a limit ϕ then $\|\phi\| \leqslant \liminf_{n\to\infty} \|u_n\|\,.$

Show that if u_n is a sequence in the Hilbert space $W^{1,2}(\mathbb{R}^3)$ which converges weakly to a limit ϕ then

$$\int_{\mathbb{R}^3} (1 - \cos\phi)\, dx \;\leqslant\; \liminf_{n\to\infty} \int_{\mathbb{R}^3} (1 - \cos u_n)\, dx\,.$$

(c) Given a function $f \in L^2(\mathbb{R}^3)$, consider the functional $E : W^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ given by

$$E[u] \;=\; \frac{1}{2} \int_{\mathbb{R}^3} \Big(|\nabla u|^2 + u^2 + 2(1 - \cos u) - 2fu \Big)\, dx\,.$$

Show that there exists a function $\phi \in W^{1,2}(\mathbb{R}^3)$ such that

$$E[\phi] = \inf_{u \in W^{1,2}(\mathbb{R}^3)} E[u]$$

and derive a partial differential equation which ϕ satisfies, giving the precise sense in which this equation holds.

Show that $\lim_{|x|\to\infty} |\phi(x)| = 0$, and that if ϕ is C^2, then

$$\sup_{x\in\mathbb{R}^3} |\phi(x)| \leqslant \sup_{x\in\mathbb{R}^3} |f(x)|\,.$$

[*Hint: You may use without proof the fact that if* $u \in W^{1,2}(\mathbb{R}^d)$ *is a weak solution of* $-\Delta u + u = G \in L^2(\mathbb{R}^d)$ *then* $\|u\|_{W^{2,2}(\mathbb{R}^d)} \leqslant C\|G\|_{L^2(\mathbb{R}^d)}$ *for some constant* $C > 0\,.$]

3 Throughout this question the time parameter t is non-negative, i.e., $t \geqslant 0$.

(a) Define what it means for the collection $\{S(t)\}_{t\geqslant 0}$ to be a *strongly continuous one-parameter semigroup of contractions* on a Banach space X, and define the *generator* A. Prove that the generator is closed and densely defined.

State the *Hille-Yosida theorem.*

Define the Sobolev space $H^1(\mathbb{R})$, and show that the space

$$X = \{u \in H^1(\mathbb{R}) : \int_{\mathbb{R}} x^2 |u(x)|^2 \, dx \, < \, \infty\}$$

is a Hilbert space with the norm given by $\|u\|_X^2 = \|u\|_{H^1}^2 + \int_{\mathbb{R}} x^2 |u(x)|^2 \, dx$. Characterize $H^1(\mathbb{R})$ as a subset of $L^2(\mathbb{R})$ in terms of the difference quotient $D_h f(x) = \frac{1}{h}(f(x+h) - f(x))$.

(b) Consider the operator $Au = \frac{d^2 u}{dx^2} - u - x^2 u$ as an unbounded linear operator on $L^2(\mathbb{R}, dx)$ with domain

$$\operatorname{Dom} A = \{u \in X \cap H^2(\mathbb{R}) : x^2 u(x) \in L^2(\mathbb{R}, dx)\}.$$

Show that A is a densely defined, closed operator, and obtain an estimate for the resolvent operator $(\lambda - A)^{-1}$ for $\lambda > 0$. Hence, deduce that the Cauchy problem for the equation

$$\partial_t \psi \, = \, \psi'' - x^2 \psi \, , \quad \psi(x, 0) = \psi_0(x) \, , \quad \psi(x, t) \in \mathbb{C}$$

determines a function $t \mapsto \psi(x,t) \in L^2(\mathbb{R}, dx)$ which is continuous into $L^2(\mathbb{R}, dx)$, satisfies $\|\psi(x,t)\|_{L^2(\mathbb{R},dx)} \leqslant e^t \|\psi_0\|_{L^2(\mathbb{R})}$, and satisfies the equation if $\psi_0 \in \operatorname{Dom} A$.

[*In this question* $L^2(\mathbb{R}, dx)$ *is sometimes written in place of* $L^2(\mathbb{R})$ *for clarity, to emphasize the relevant independent variable.*]

END OF PAPER