UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2016 1:30 pm to 3:30 pm

PAPER 104

INFINITE GROUPS AND DECISION PROBLEMS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Given two groups G_1 and G_2, define the *free product* $G_1 * G_2$. What is meant by a *non trivial* free product?

State and prove the normal form theorem for free products and the universal property of free products.

Which of the following groups can be expressed as non trivial free products? (Give brief reasons for your answers.)

(i) The alternating group A_5.

(ii) The group $\mathbb{Z}^4$.

(iii) The group G defined by the presentation

$$\langle x, s, t \mid x^2, xsx^{-1} = t, xtx^{-1} = s \rangle.$$

2 Define a *graph* (a 1-dimensional CW complex) and a *subgraph.*

Show that given any connected graph Γ there exists a subgraph Δ containing all the vertices of Γ and such that Δ is homotopy equivalent to a point.

Show that a covering space of a connected graph also has the structure of a graph. By assuming that a connected finite graph Γ has fundamental group $\pi_1(\Gamma)$ which is free with rank the number of edges in $\Gamma \backslash \Delta$ for Δ as above, state and prove the Nielsen–Schreier index formula.

Can the free group F_n of rank $n \geqslant 1$ be isomorphic to a proper finite index subgroup of itself? [You may assume that $F_n \cong F_m$ exactly when $n = m$.]

3 Given a group G with subgroups A, B and $\phi : A \to B$ an isomorphism, define the *HNN extension* $G*_\phi$ and what is meant by a *normal form.*

Show that every element in $G*_\phi$ has a unique normal form.

Define a *reduced* sequence. State and prove Britton's lemma.

Now set $G = \langle a \rangle \cong \mathbb{Z}$, $A = \langle a^2 \rangle$ and $B = \langle a^3 \rangle$ with $\phi(a^2) = a^3$ in the above to form the HNN extension $S = G*_\phi$ with stable letter t, which you may assume is defined by the presentation

$$\langle a, t \,|\, ta^2t^{-1} = a^3 \rangle.$$

(i) Show that the homomorphism $\theta : S \to S$ given by $\theta(\overline{a}) = \overline{a}^2$ and $\theta(\overline{t}) = \overline{t}$ is well defined.

(ii) Show also that θ is a surjective homomorphism.

(iii) Explain why the element $[\overline{a}, \overline{t}\overline{a}\overline{t}^{-1}]$, where $[x, y] = xyx^{-1}y^{-1}$, is mapped to the identity in S by θ.

(iv) Establish that $[\overline{a}, \overline{t}\overline{a}\overline{t}^{-1}]$ is not the identity in S, and thus we have a surjective homomorphism from a finitely presented group to itself which is not injective.

4

Let T be a Turing machine with alphabet $S = \{s_0, \ldots, s_m\}$, states $Q = \{q_0, \ldots, q_n\}$, initial state q_1, and halting state q_0.

(a) Write down the finite presentation $\Gamma(T)$ for the *associated semigroup* of T.

(b) Let T and $\Gamma(T)$ be as above, and let $\Omega(T)$ denote the *halting set* of T. Show that, for any $w \in S^+$, we have that

$$w \in \Omega(T) \text{ if and only if } \overline{hq_1wh} = \overline{q} \text{ in } \overline{\Gamma(T)}.$$

5

(a) Let A, B be groups. Show that (a, b) has finite order in $A \times B$ if and only if both a and b have finite order in A and B respectively.

(b) Let $G*_{\phi}$ be an HNN extension with base group G, and let $\gamma \in G*_{\phi}$. Show that γ has finite order if and only if γ is conjugate to an element in G of finite order.

Let $\mathcal{M} = \{(a_i, b_i, c_i, R) \mid i \in I\} \cup \{(a_j, b_j, c_j, L) \mid j \in J\}$ be a modular machine with modulus m, in which $(0, 0)$ is terminal and for which $H_0(\mathcal{M})$ is not recursive.

(c) Define the free product K via a presentation, and the words $t(r, s)$ in this presentation (for $r, s \in \mathbb{Z}$), as in the construction of a finitely presented group with unsolvable word problem.

(d) For $M > a \geqslant 0$ and $N > b \geqslant 0$, define the subgroup $K_{a,b}^{M,N} \leqslant K$, and for all $i \in I$ and $j \in J$ define the isomorphisms

$$\phi_i : K_{a_i,b_i}^{m,m} \to K_{c_i,0}^{m^2,1}$$

$$\varphi_j : K_{a_j,b_j}^{m,m} \to K_{0,c_j}^{1,m^2}$$

(e) Define the HNN extension $K_{\mathcal{M}}$, and the subgroup $\langle \bar{t} \rangle' \leqslant K_{\mathcal{M}}$. State the relationship between $H_0(\mathcal{M})$ and $\langle \bar{t} \rangle'$.

(f) Define the HNN extension $G_{\mathcal{M}}$, and prove that the group $G_{\mathcal{M}}$ has unsolvable word problem.

(g) Prove that the group $G_{\mathcal{M}}$ is torsion free.

6

(a) Give an algorithmic construction that, on input of a finite presentation $P = \langle X|R\rangle$ of a group, and a word $w \in F(X)$, outputs a finite presentation $P(w)$ of a group with the following properties:

1. If $\overline{w} = e$ in $\overline{P}$, then $\overline{P(w)} = \{e\}$.
2. If $\overline{w} \neq e$ in $\overline{P}$, then $\overline{P}$ embeds in $\overline{P(w)}$.

(b) Define what it means for an algebraic property ρ of finitely presented groups to be a *Markov property*. Show that it is impossible to algorithmically recognise any Markov property amongst finitely presented groups. [You may assume the existence of a finitely presented group with unsolvable word problem.]

(c) Consider Higman's group with finite presentation

$$Q := \langle a_1, a_2, a_3, a_4 \mid a_1a_2a_1^{-1} = a_2^2,\ a_2a_3a_2^{-1} = a_3^2,\ a_3a_4a_3^{-1} = a_4^2,\ a_4a_1a_4^{-1} = a_1^2\rangle$$

Using this, and your results in earlier parts of this question, show that each finitely presented group embeds into some finitely presented group with no non-trivial finite quotients. [You may state without proof any properties of Higman's group that you use.]

(d) Hence show that neither of the following are Markov properties:

(i) Having at least one non-trivial finite quotient.

(ii) Having no non-trivial finite quotients.

END OF PAPER