UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 2 June, 2016 1:30 pm to 4:30 pm

PAPER 103

REPRESENTATION THEORY

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $Z_n = Z(\mathbb{C}S_n)$, the centre of the symmetric group algebra. Define

(a) the *Young-Jucys-Murphy elements* X_i (for $i = 1, 2, \ldots, n$);

(b) the *Gelfand-Tsetlin algebra* GZ_n associated to an inductive family of subgroups of S_n;

(c) the *centraliser algebra* $Z_{(n-1,1)}$.

Stating clearly any results you need, show that

(i) $Z_{(n-1,1)}$ is generated by Z_{n-1} and X_n and

(ii) GZ_n is generated by $X_1, \ldots, X_n$.

Deduce that the inductive chain $S_1 \leqslant \cdots \leqslant S_n$ defines simple branching, in the sense that the multiplicities of the restrictions of irreducible representations of $\mathbb{C}S_j$ to $\mathbb{C}S_{j-1}$ equal 0 or 1 (semisimplicity results may be assumed, if clearly stated).

Now let $V := \{(z_1, \ldots, z_n) : z_1 + \cdots z_n = 0\}$ and let S_n act on V by permutations of the coordinates. You may assume V is irreducible for this action.

(iii) Pick a basis $\{v_j\}$ of V and compute $X_i v_j$ for all i, j.

(iv) For $n = 4$ find the Gelfand-Tsetlin basis of V.

2

Define the Young poset $\mathbb{Y}$. Let Y_i be the ith level of $\mathbb{Y}$, namely the set of all partitions of i (including the 0th level with unique partition $\emptyset$). Define also the Young (branching) graph. Briefly explain a correspondence between paths in the Young graph from a given vertex λ to (1) and standard Young tableaux of shape λ.

Let $\mathbb{R}\mathbb{Y}$ be the space of all formal (finite) linear combinations of Young diagrams, including the empty diagram and let $\mathbb{R}Y_i$ be the subspace with basis Y_i. For all $\lambda \vdash i$, define two linear transformations $U_i : \mathbb{R}Y_i \to \mathbb{R}Y_{i+1}$ and $D_i : \mathbb{R}Y_i \to \mathbb{R}Y_{i-1}$, by $U_i(\lambda) = \sum \mu$, where the sum is over all $\mu \vdash i+1$ and μ covers λ and $D_i(\lambda) = \sum \nu$ where the sum is over all $\nu \vdash i-1$ and λ covers ν.

(a) Show that for any $i \geqslant 0$,

$$D_{i+1}U_i - U_{i-1}D_i = I_i,$$

the identity linear transformation on $\mathbb{R}Y_i$.

(b) For notational simplicity omit the subscripts from the linear transformations U_i and D_i (since the subscripts will be uniquely determined by the elements on which U and D act). Let $\lambda \vdash n$. Let $N(\ell, \lambda)$ denote the number of (upward) paths of length ℓ from the empty partition to λ, and it is given that this number is the (well-defined) coefficient of λ in the expansion of $(D+U)^\ell(\emptyset)$ as a linear combination of partitions. Define integers $a_{i,j}(\ell)$ by

$$(D+U)^\ell = \sum_{i,j} a_{i,j}(\ell) \quad U^i D^j.$$

Show that $a_{i,j}(\ell) = 0$ if $\ell - i - j$ is odd, while if $\ell - i - j = 2m$,

$$a_{i,j}(\ell) = \frac{\ell!}{2^m i! j! m!}.$$

[Hint: use induction on ℓ.]

Deduce that, if $\ell \geqslant n$ and $\lambda \vdash n$, with $\ell - n$ even,

$$N(\ell, \lambda) = \binom{\ell}{n} (1 \cdot 3 \cdot 5 \cdots (\ell - n - 1)) f_\lambda,$$

and hence derive an expression for $N(2m, \emptyset)$.

3

Let $G = S_n$. Define the *spectrum* Spec(n) of the Young-Jucys-Murphy elements of G. Define the *Young basis* of an irreducible representation of G. What is the *weight* of any basis vector? Define the set Cont(n) of *content vectors* of length n.

(i) Let $\alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. If $a_i = a_{i+2} = a_{i+1} - 1$ for some $i \in \{1, 2, \ldots, n-2\}$, show that $\alpha \notin$Spec(n).

(ii) Explain why, for every $(a_1, \ldots, a_n) \in$Spec(n) we have $a_1 = 0$. Compute Spec(2) and Cont(2).

(iii) Define the equivalence relations $\sim$ and $\approx$ on Spec(n) and Cont(n), respectively.

(iv) For every $n \geqslant 1$ show that Spec(n) is contained in Cont(n).

(v) Show that Spec(n) = Cont(n) and that the equivalence relations $\sim$ and $\approx$ coincide.

4 Let $\lambda \vdash n$. If $x = (i, j)$ is a box in the Young diagram of λ, define the *hook*, Γ_x with corner x and define the corresponding *hook-length* $h(x)$. State the hook-lengths formula for f_λ (the number of standard tableaux of a given shape λ).

(a) In the hook walk process determined by Greene, Nijenhuis and Wilf, find a closed expression for the probability $p(\alpha, \beta \mid a, b)$ that a box (α, β) will be the terminal box, given that (a, b) is the initial box.

(b) (i) Let $k \in \{1, \ldots, n\}$. Prove that the product $\xi_k := X_2 \ldots X_k$ of YJM elements equals the sum of all k-cycles in S_k.

(ii) Find the eigenvalues of ξ_k on any vector of the Young basis in S^λ. [Hint: consider separately the cases when λ is a hook and when λ is not a hook (recall that λ is called a *hook* if it is of the form $(a+1, 1^b)$ for some non-negative integers a, b).]

(iii) Defining χ^λ to be the character of S^λ, evaluate χ^λ on the k-cycle $(1\ 2\ \ldots\ k)$.

(c) If λ has k non-zero parts, prove the *determinantal formula*

$$f_\lambda = n! \det(1/(\lambda_i - i + j)!),$$

where the determinant is that of a $k \times k$ matrix. By convention we set $1/r! = 0$ if $r < 0$. [Hint: show that the denominator yields the denominator of the hook-lengths formula.]

5

Let $\lambda \vdash n$. If T is a standard tableau of shape λ, define the *content*, $C(T)$, of T. If $i, j \in \{1, 2, \ldots, n\}$, define the *axial distance* from j to i in T.

Define and prove *Young's orthogonal form* for the orthonormal basis $\{w_T : T \in \mathrm{Tab}(n)\}$. Results about Young's seminormal form may be assumed provided they are clearly stated.

For each of the following partitions λ of n, list the standard λ-tableaux T, give the corresponding content $C(T)$ and find a formula for the action of each Coxeter element s_j on the vectors w_T:

(i) $\lambda = (n)$;

(ii) $\lambda = (1^n)$;

(iii) $\lambda = (n-1, 1)$.

In each case identify the relevant Specht module, S^λ. You do not need to give detailed proofs of the identifications.

6

Describe the process of *row insertion* $R \leftarrow x$ of $x \notin R$ into a near Young tableau (NYT) R. Briefly describe the Robinson-Schensted correspondence $\pi \mapsto (P, Q)$ between elements $\pi \in S_n$ and pairs (P, Q) of standard tableaux of the same shape.

(a) Let P be a NYT, with $x \notin P$. Let $P_{i,j}$ stand for the (i, j) entry of P. In the usual notation, suppose that during the row insertion $r_x(P) = P'$, the elements $x', x'', x''', \ldots$ are bumped from boxes $(1, j'), (2, j''), (3, j'''), \ldots$, respectively. Show that

(i) $x < x' < x'' < \cdots$

(ii) $j' \geqslant j'' \geqslant j''' \geqslant \cdots$

(iii) $P'_{i,j} \leqslant P_{i,j}$ for all i, j.

Given $\pi = x_1 x_2 \ldots x_n \in S_n$, an *increasing subsequence* of π is

$$x_{i_1} < x_{i_2} < \cdots < x_{i_k}$$

where $i_1 < i_2 < \cdots < i_k$. We call the integer k the *length* of the subsequence.

Let P_{k-1} be the tableau formed after $k - 1$ insertions of the Robinson-Schensted procedure. Show, by induction on k, or otherwise, that if $\pi = x_1 x_2 \ldots x_n$ and x_k enters P_{k-1} in column j, then the longest increasing subsequence of π ending in x_k has length j.

Deduce that the length of the longest increasing subsequence of π is the length of the first row of the insertion tableau $P(\pi)$.

(b) What is a *generalised permutation*? Consider the set $\mathbb{P}$ of generalised permutations where no column is repeated. Assuming Knuth's generalisation of the Robinson-Schensted correspondence, show that there is a bijection between $\pi \in \mathbb{P}$ and pairs (T, U) of tableaux of the same shape, with T and U^t semistandard (you should describe the map $\pi \to (P, Q)$ but need not verify explicitly it has an inverse). Briefly explain how the type of T and U can be calculated from π.

END OF PAPER