UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 30 May, 2016 9:00 am to 12:00 pm *Draft 4 July, 2016*

PAPER 102

LIE ALGEBRAS AND THEIR REPRESENTATIONS

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

All Lie Algebras are complex and finite dimensional unless otherwise stated.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
Triangular Graph Paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove Lie's theorem.

Let k be a field of characteristic p. Consider the two $p \times p$ matrices over k

$$x := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad y := \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & p-2 & 0 \\ 0 & 0 & \dots & 0 & p-1 \end{pmatrix}$$

Show that x and y have no common eigenvector.

Calculate $[x, y]$ and deduce that the conclusion of Lie's theorem does not hold over k.

Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$ for V finite-dimensional and complex. Show that $\mathcal{D}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

You may use a standard result if correctly identified.

Deduce that any complex Lie algebra $\mathfrak{g}$ is solvable if and only if $\mathcal{D}(\mathfrak{g})$ is nilpotent.

2

Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie algebra. What is the Killing form with respect to V on $\mathfrak{g}$? State its important properties.

Let $\mathfrak{g}$ be semisimple and $\mathfrak{h}$ a Cartan subalgebra. For each root $\alpha \in \mathfrak{h}^*$, prove the existence of an $\mathfrak{sl}(2)$-subalgebra $\mathfrak{s}_\alpha = \langle X_\alpha, H_\alpha, Y_\alpha \rangle$ with $X_\alpha \in \mathfrak{g}_\alpha$, $H_\alpha \in \mathfrak{h}$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$, satisfying $[H_\alpha, X_\alpha] = 2X_\alpha$, $[H_\alpha, Y_\alpha] = -2Y_\alpha$, $[X_\alpha, Y_\alpha] = H_\alpha$.

You may assume that the restriction of the (adjoint) Killing form to $\mathfrak{g}$ is non-degenerate. You may use Lie's theorem.

You may also assume that the restriction of the Killing form to $\mathfrak{h}$ is non-degenerate.

Define the weight lattice and explain briefly why the weights of any finite-dimensional representation must lie in the weight lattice.

You may use any facts about finite-dimensional representations of $\mathfrak{sl}(2)$ provided these are stated clearly.

Define the Lie algebra $\mathfrak{sp}(4)$, give a basis of a Cartan subalgebra and a Cartan decomposition. In terms of those basis elements give bases for each subalgebra $\mathfrak{s}_\alpha$ where α is a positive root.

3

State and prove Schur's Lemma.

State and prove Weyl's theorem on complete reducibility.

You may use any properties of the Casimir operator if clearly stated. You may assume that semisimple Lie algebras act trivially on 1-dimensional representations.

Let $\mathfrak{g}$ be a Lie algebra. What is a derivation of $\mathfrak{g}$?

Explain how one can put a structure of a Lie algebra on the set of all derivations of $\mathfrak{g}$.

Prove that $\mathrm{ad}(\mathfrak{g})$ is an ideal of $\mathrm{Der}(\mathfrak{g})$.

Assuming $\mathfrak{g}$ is semisimple, show that all derivations of $\mathfrak{g}$ are inner, i.e., show that $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$.

Hint: You may wish to use Weyl's theorem for this last part.

4

If V is a finite-dimensional representation of a Lie algebra $\mathfrak{g}$ over a field k with basis $\{v_1, \ldots, v_n\}$, then construct the representations $\bigwedge^2 V$ and S^2V of $\mathfrak{g}$.

Prove that if k is a field of odd characteristic, then $V \otimes V \cong \bigwedge^2 V \oplus S^2 V$.

For any two finite-dimensional representations V and W of a Lie algebra $\mathfrak{g}$, prove that

$$S^2(V \oplus W) \cong S^2V \oplus S^2W \oplus (V \otimes W). \qquad (*)$$

Let $\mathfrak{g} = \mathfrak{sl}(3)$, let $V = \Gamma_{0,0} \cong \mathbb{C}$ be the trivial representation and $W = \Gamma_{2,1}$. Calculate the decomposition into irreducibles of

$$S^2(V \oplus W).$$

Hint: You are encouraged to check your answer using Weyl's dimension formula, which for $\mathfrak{sl}(3)$ reads

$$\dim \Gamma_{a,b} = \frac{(a+1)(b+1)(a+b+2)}{2}$$

5

Define an (abstract) root system.

Establish the possible angles between any two roots.

Prove that there is only one irreducible root system in which some pair of simple roots has an angle of $5\pi/6$ between them.

You may assume the existence of a set of simple roots, that the Weyl group is generated by reflections in the simple roots, and that all roots are conjugate to simple roots under the Weyl group.

6

Explain what is meant by a vector field on a manifold.

What does it mean for a vector field on a Lie group G to be left-invariant?

Given an element $X \in T_g(G)$ explain how one gets a vector field on G associated to X.

Prove that if $X \in T_g(G)$ with v_X the associated vector field then there exists a unique curve $\gamma_g(t) : \mathbb{R} \to G$ such that $\gamma_g(0) = g$ and $\gamma_g'(t) = v_X(\gamma(t))$.

Prove that for any Lie group G the identity component G° is an open normal subgroup of G. Moreover, if U is any open neighbourhood of the identity in G° then G° is generated by U.

What is a quadratic form Q on a vector space $V = \mathbb{R}^n$?

What does it mean for an element $g \in \mathrm{GL}_n(\mathbb{R})$ to stabilise a quadratic form?

Find a description of the matrices in $T_e(G)$ for the subgroup $G = G_Q$ of $\mathrm{GL}_n(\mathbb{R})$ stabilising the quadratic form Q.

END OF PAPER