UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 27 May, 2016 1:30 pm to 4:30 pm *Draft 4 July, 2016*

PAPER 101

COMMUTATIVE ALGEBRA

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $R = k[X_1, X_1^{-1}, X_2, X_2^{-1}]$ where k is a field.

Show that R is Noetherian.

For a_1 and a_2 in $\mathbb{Z}$, let $\theta_{(a_1,a_2)} : R \longrightarrow k[T, T^{-1}]$

$\sum_{m_1,m_2} \lambda_{m_1,m_2} X_1^{m_1} X_2^{m_2} \mapsto \sum_{m_1,m_2} \lambda_{m_1,m_2} T^{a_1 m_1 + a_2 m_2}$

where m_1 and m_2 are in $\mathbb{Z}$ and λ_{m_1,m_2} is in k.

Show that if r is a non-zero element of R then there exists some $\theta_{(a_1,a_2)}$ such that $\theta_{(a_1,a_2)}(r)$ is non-zero.

Show that any k-subalgebra of $k[T, T^{-1}]$ is Noetherian.

Give an example of a non-Noetherian of R.

2

State and prove the weak Nullstellensatz

Let $R = \mathbb{R}[X_1, X_1^{-1}, X_2, X_2^{-1}]$.

Describe the maximal ideals of R.

Show that any prime ideal P of R is the intersection of the maximal ideals containing P.

Define the set of associated primes of a non-zero finitely generated R-module M. Show that this set is non-empty.

What does it mean for an ideal of R to be P-primary?

Give an example of an ideal I of R and a prime ideal P such that P is the only minimal prime over I but I is not P-primary.

3

Let $R = \mathbb{Z}[X_1, X_1^{-1}, X_2, X_2^{-1}]$ and let $\theta : R \longrightarrow \mathbb{Z}$,

$\sum_{m_1,m_2} \lambda_{m_1,m_2} X_1^{m_1} X_2^{m_2} \mapsto \sum_{m_1,m_2} \lambda_{m_1,m_2}$

where m_1, m_2 and λ_{m_1,m_2} are in $\mathbb{Z}$.

Denote the kernel of θ by P.

Show that P/P^2 is isomorphic to $\mathbb{Z}^2$.

Show that the localisation R_P of R at P is of dimension 2.

Define what it means for a local ring to be regular. Show that R_P is a regular local ring.

4

Define what it means for an integral domain to be integrally closed.

Show that any unique factorisation domain is integrally closed.

State and prove the Going Down theorem.

5

Let R be a Noetherian ring, and let I be an ideal. Let M be a finitely generated R-module with submodule N.

State and prove the Artin-Rees lemma.

Show that $\cap_{j=1}^{\infty} I^j M$ consists of those m in M that are annihilated by some element of the form $1 + r$ with $r \in I$.

Give an example of a non-Noetherian ring R, an ideal I and an element $m \in \cap_{j=1}^{\infty} I^j$ which is not annihilated by any element of the form $1 + r$ with $r \in I$.

6

Let k be a field and S be the polynomial ring $k[X_1, \ldots, X_n]$ considered as a graded ring, graded by total degree.

State and prove the Hilbert-Serre theorem concerning the Poincare series of a finitely generated graded S-module V.

Sketch a proof that there is a free resolution of V of length at most n. (You should explain any terminology that you use.)

END OF PAPER