UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2015 9:00 am to 12:00 pm

PAPER 9

ELLIPTIC PDES

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let Ω be a bounded domain in $\mathbb{R}^n$ and

$$Lu = D_i(a^{ij}D_ju + b^iu) + c^iD_iu + du$$

be an elliptic operator in divergence form with coefficients $a^{ij} \in L^\infty(\Omega)$, $b^i, c^i \in L^q(\Omega)$, and $d \in L^{q/2}(\Omega)$, where $q > n$.

(i) State, without proof, the weak Harnack inequality for supersolutions $u \in W^{1,2}(\Omega)$ satisfying $Lu \leqslant D_if^i + g$ weakly in Ω for $f^i \in L^q(\Omega)$ and $g \in L^{q/2}(\Omega)$.

(ii) Deduce from Part (i) the following version of the strong maximum principle: Let $b^i = d = 0$ a.e. in Ω. Suppose $u \in W^{1,2}(\Omega)$ such that $Lu \geqslant 0$ weakly in Ω. If there is an open ball $B \subset\subset \Omega$ such that

$$\sup_B u = \sup_\Omega u,$$

then u is constant on Ω.

(iii) Deduce from Part (i) the following: Suppose $u \in W^{1,2}(\Omega)$ such that $Lu = D_if^i + g$ weakly in Ω for some $f^i \in L^q(\Omega)$ and $g \in L^{q/2}(\Omega)$. Then

$$[u]_{\mu;\Omega'} \leqslant C\left(\sup_\Omega |u| + \|f^i\|_{L^q(\Omega)} + \|g\|_{L^{q/2}(\Omega)}\right)$$

for all $\Omega' \subset\subset \Omega$ and some constant $C = C(n, L, \Omega', \Omega) \in (0, \infty)$.

2

Consider the partial differential equation

$$Lv = a^{ij} D_{ij} v + b^i D_i v + cv = f$$

in $\Omega \subset \mathbb{R}^n$, where a^{ij}, b^i, c, f are bounded functions on Ω, $c \leqslant 0$ and a^{ij} satisfy the ellipticity condition $a^{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2$ for all $x \in \Omega$ and $xi \in \mathbb{R}^n$ and some constant $\lambda > 0$.

(a) State and prove the weak maximum principle for $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

(b) Prove that there exists at most one solution to the Dirichlet problem

$$\begin{cases} a^{ij} D_{ij} u + b^i D_i u + cu = f \text{ in } \Omega \\ u|_{\partial\Omega} = \varphi \in C^0(\partial\Omega) \end{cases}$$

(c) Show by means of an example that for $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Lu \geqslant 0$ the inequality $\sup_\Omega u \leqslant \sup_{\partial\Omega} u$ might fail.

3

Let $B \subset \mathbb{R}^n$ be the open unit ball and let $u \in W^{1,2}(B)$ solve the partial differential equation

$$D_i(a^{ij} D_j u) = 0,$$

weakly in B, where $a^{ij} \in L^\infty(B)$ (say $|a^{ij}| \leqslant \Lambda$) satisfy the ellipticity condition $a^{ij}\xi_i\xi_j \geqslant \lambda|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$ and some constant $\lambda \in (0,\infty)$.

(a) Prove that for any $c \in \mathbb{R}$, for any $x_0 \in B$ and $R > 0$ such that $B_{2R}(x_0) \subset B$ it holds

$$\int_{B_R(x_0)} |\nabla u|^2 \leqslant \frac{C(\lambda,\Lambda)}{R^2} \int_{B_{2R}(x_0)\setminus B_R(x_0)} |u-c|^2.$$

(b) Prove that there exist constants $\mu \in (0,1)$ and $K \in (0,\infty)$ independent of u and x_0 such that whenever $B_{r_0}(x_0) \subset B$ we have the following estimate for all $r \leqslant r_0$:

$$\int_{B_r(x_0)} |\nabla u|^2 \leqslant K\left(\frac{r}{r_0}\right)^\mu \int_B |\nabla u|^2.$$

[HINT: you may use without proof the Poincaré inequality

$$\int_{B_{2R}(x_0)\setminus B_R(x_0)} |v-\overline{v}|^2 \leqslant C(n)R^2 \int_{B_{2R}(x_0)\setminus B_R(x_0)} |\nabla v|^2,$$

where $\overline{v}$ stands for the average of v in the annulus $B_{2R}(x_0) \setminus B_R(x_0)$.]

4

(a) Show that if $u \in C^{2,\mu}(\overline{B_1(0)})$, where $\mu \in (0,1)$, is a solution to

$$\Delta u = f \text{ in } B_1(0)$$

for some $f \in C^{0,\mu}(\overline{B_1(0)})$, then for every $\delta > 0$,

$$[D^2 u]_{\mu;B_{1/2}(0)} \leqslant \delta [D^2 u]_{\mu;B_1(0)} + C\left(\|u\|_{C^2(B_1(0))} + \|f\|_{C^{0,\mu}(B_1(0))}\right)$$

for some constant $C = C(n,\mu,\delta) \in (0,\infty)$.

[You may assume without proof Liouville's lemma: There does not exist any nonconstant harmonic function $w \in C^0(\mathbb{R}^n)$ such that $[w]_{\mu;\mathbb{R}^n} < \infty$.]

(b) Assuming standard elliptic theory without proof, show that the set

$$S = \left\{ u \in C^2(\overline{B_1(0)}) : \|\Delta u\|_{C^{0,\mu}(B_1(0))} \leqslant 1,\ u = 0 \text{ on } \partial\Omega \right\},$$

where $\mu \in (0,1)$, is sequentially compact in $C^2(\overline{B_1(0)})$.

5

Consider the rectangle $R = \{0 < x < \pi, 0 < y < 1\} \subset \mathbb{R}^2$ and let $u \in C^0(\overline{R}) \cap C^2(R)$ be a solution of the partial differential equation

$$u_{xx} + y^2 u_{yy} = 0.$$

(a) Assume that $u(0, y) = u(\pi, y) = 0$ for $0 \leqslant y \leqslant 1$. Prove that u must be of the form

$$\sum_{n \in \mathbb{Z}} A_n f_n(y) \sin(nx),$$

where

$$f_n(y) = y^{\frac{1}{2}(1+\sqrt{1+4n^2})}$$

and $A_n \in \mathbb{R}$ are constants.

(b) Consider the Dirichlet problem

$$\begin{cases} u_{xx} + y^2 u_{yy} = 0 \text{ in } R \\ u|_{\partial R} = \varphi \in C^0(\partial R). \end{cases} \tag{$\star$}$$

Prove that there exist choices of φ for which no solution to $(\star)$ exists.

(c) Under suitable assumptions, the Dirichlet Problem for an elliptic partial differential equation on a bounded domain Ω with continuous boundary datum does admit a solution. Which assumption is not satisfied in $(\star)$?

6

Let Ω be a bounded domain in $\mathbb{R}^n$ and consider the differential equation

$$\Delta u = \sum_{i=1}^{n} D_i(b^i u) + cu + f \text{ in } \Omega, \qquad (\star)$$

where $b^i, c, f \in C^\infty(\Omega)$.

(a) Formulate what it means for $u \in L^2(\Omega)$ to be a weak solution to $(\star)$ in Ω.

(b) Let $\{\phi_\sigma\}_{\sigma>0}$ be a family of smooth mollifiers $\phi_\sigma : \mathbb{R}^n \to [0, \infty)$ with $\operatorname{spt} \phi_\sigma \subset B_\sigma$ and $\int_{B_\sigma(0)} \phi_\sigma = 1$. Given a function $v \in L^1(\Omega)$ and $\sigma > 0$, let v_σ denote the convolution of v with ϕ_σ.

Show that if $u \in L^2(\Omega)$ is a weak solution to $(\star)$ then

$$\Delta u_\sigma(x) = \sum_{i=1}^{n} D_i(b^i u)_\sigma(x) + (cu)_\sigma(x) + f_\sigma(x)$$

for all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > \sigma$.

(c) Use Part (b) and standard elliptic estimates to show that whenever $u \in L^2(\Omega)$ is a weak solution to $(\star)$, $u \in C^\infty(\Omega)$.

END OF PAPER