UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 9 June, 2015 1:30 pm to 4:30 pm

PAPER 81

QUANTUM CONDENSED MATTER FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The one-dimensional spin S quantum Heisenberg antiferromagnet is specified by the Hamiltonian,

$$\hat{H} = J \sum_{n=1}^{N} \hat{\mathbf{S}}_n \cdot \hat{\mathbf{S}}_{n+1} ,$$

where $J > 0$, the number of lattice sites N is even, and periodic boundary conditions are imposed such that $\hat{\mathbf{S}}_{N+1} \equiv \hat{\mathbf{S}}_1$.

(a) Define the classical ground state degeneracy of the system and specify one of the possible classical antiferromagnetic ground states. Explain why the classical ground state is not an eigenstate of the quantum Hamiltonian.

(b) Applying a suitable canonical transformation, show that the Hamiltonian can be brought to the form

$$\hat{H} = -J \sum_{n=1}^{N} \left[\hat{S}_n^z \hat{S}_{n+1}^z - \frac{1}{2} \left(\hat{S}_n^+ \hat{S}_{n+1}^+ + \hat{S}_n^- \hat{S}_{n+1}^- \right) \right] ,$$

where $\hat{S}_n^{\pm} = \hat{S}_n^x \pm i \hat{S}_n^y$ denote spin raising and lowering operators.

(c) Show that the Holstein-Primakoff transformation,

$$\hat{S}^+ = (2S)^{1/2} \left(1 - \frac{a^\dagger a}{2S} \right)^{1/2} a, \qquad \hat{S}^- = (\hat{S}^+)^\dagger, \qquad \hat{S}^z = S - a^\dagger a ,$$

is consistent with the quantum spin algebra, $[\hat{S}^+, \hat{S}^-] = 2\hat{S}^z$ (where we have set $\hbar = 1$).

(d) Taking the spin to be large $S \gg 1$, show that to first order in S, the antiferromagnetic Hamiltonian can be written as a bilinear in lattice boson operators a_n and $a_n^\dagger$.

(e) Bringing the Hamiltonian to diagonal form, show that the excitation spectrum of the resulting Hamiltonian is given by $\omega_k = 2JS|\sin k|$. Define the quantum numbers k and comment on the form of the low-energy spectrum.

(f) Show that the ground state sublattice magnetisation is given by

$$\langle \text{g.s.} | \frac{1}{N} \sum_{n=1}^{N} (-1)^n \hat{S}_n^z | \text{g.s.} \rangle \simeq S - \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{1}{2} \left[\frac{1}{|\sin k|} - 1 \right] .$$

Comment on the physical implications of this result.

2

In the BCS approximation, the pairing Hamiltonian of a superconducting electron system is given by

$$\hat{H} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} b^\dagger_{\mathbf{k}} b_{\mathbf{k}'} \,,$$

where $b^\dagger_{\mathbf{k}} \equiv c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}$, and the fermion operators $c^\dagger_{\mathbf{k}\sigma}$ and $c_{\mathbf{k}\sigma}$ respectively create and annihilate electrons with wavevector $\mathbf{k}$ and spin σ.

(a) Neglecting terms of order $(b_{\mathbf{k}} - \langle b_{\mathbf{k}} \rangle)^2$, where $\langle b_{\mathbf{k}} \rangle \equiv \langle \text{g.s.} | b_{\mathbf{k}} | \text{g.s.} \rangle$ denotes the ground state expectation value, show that

$$\hat{H} - \mu \hat{N} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \left(\Delta_{\mathbf{k}} b^\dagger_{\mathbf{k}} + \text{h.c.} \right) - \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} \langle b^\dagger_{\mathbf{k}} \rangle \langle b_{\mathbf{k}'} \rangle \,,$$

where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$ and $\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} \langle b_{\mathbf{k}'} \rangle$.

(b) Under what conditions on the real coefficients $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ is the following transformation canonical,

$$\begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} u_{\mathbf{k}} & v_{\mathbf{k}} \\ -v_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} \gamma_{\mathbf{k}\uparrow} \\ \gamma^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix}.$$

(c) Taking $\Delta_{\mathbf{k}}$ to be real, use this transformation to show that the BCS pairing Hamiltonian can be brought to the diagonal form,

$$\hat{H} - \mu \hat{N} = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}\sigma} \gamma_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \left(\xi_{\mathbf{k}} - E_{\mathbf{k}} + \Delta_{\mathbf{k}} \langle b^\dagger_{\mathbf{k}} \rangle \right) ,$$

where $E_{\mathbf{k}} = (\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2)^{1/2}$. Show that the corresponding ground state wavefunction is given by

$$|\text{g.s.}\rangle = \prod_{\mathbf{k}} \left(\cos\theta_{\mathbf{k}} + \sin\theta_{\mathbf{k}}\, c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right) |0\rangle \,,$$

with $|0\rangle$ the vacuum state of the electron system.

(d) At zero temperature, show that

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V_{\mathbf{k},\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}}.$$

Taking $V_{\mathbf{k},\mathbf{k}'} = -V/L^3$ for $\{|\xi_{\mathbf{k}}|, |\xi_{\mathbf{k}'}|\} < \hbar\omega_{\text{D}}$ and zero otherwise, show that $\Delta_{\mathbf{k}} = \Delta \delta_{\mathbf{k},0}$, where

$$\Delta = \frac{\hbar\omega_D}{\sinh(1/\nu V)},$$

and ν is the total electronic density of states at the Fermi level.

You may note the identity $\int_0^z dx\, (1+x^2)^{-1/2} = \sinh^{-1}(z)$.

3

Without detailed derivation, state the Lagrangian form of the Feynman path integral for the Hamiltonian,

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(q), \qquad \hat{p} = -\mathrm{i}\hbar\partial_q.$$

(a) Defining the eigenstates of a free quantum particle confined to an infinite square well potential of width L (centred on $q = L/2$), show that the quantum partition function, $\mathcal{Z} = \mathrm{tr}\, \mathrm{e}^{-\beta\hat{H}}$, is given by the sum

$$\mathcal{Z} = \sum_{n=1}^{\infty} \exp\left[-\beta\frac{(\hbar\pi n)^2}{2mL^2}\right].$$

(b) Starting with the Lagrangian form of the Feynman path integral, show that the quantum partition function of a particle in the infinite square well potential can be also be expressed as

$$\mathcal{Z} = \int_0^L dq \sum_{r=-\infty}^{\infty} \left[\int_{q(0)=q,\, q(\beta)=2rL+q} Dq(\tau)\, \mathrm{e}^{-S(q)/\hbar} - \int_{q(0)=q,\, q(\beta)=2rL-q} Dq(\tau)\, \mathrm{e}^{-S(q)/\hbar}\right],$$

where $S(q) = \dfrac{1}{\hbar}\displaystyle\int_0^\beta d\tau\, \frac{1}{2}m(\partial_\tau q)^2$.

Hint: To obtain this result, you will find it useful to draw a figure showing typical Feynman trajectories, and note that each scattering event from a wall of the infinite potential well is accompanied by a π phase shift of the Feynman amplitude.

(c) By making use of the expression for the free particle propagator, $\langle q|\exp\left[-\dfrac{\mathrm{i}}{\hbar}\dfrac{\hat{p}^2}{2m}t\right]|q\rangle = \left(\dfrac{m}{2\pi\mathrm{i}\hbar t}\right)^{1/2}$, evaluate the path integral above and show that

$$\mathcal{Z} = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{1/2} \int_0^L dq \sum_{r=-\infty}^{\infty} \left(\exp\left[-\frac{2m}{\hbar^2}\frac{(rL)^2}{\beta}\right] - \exp\left[-\frac{2m}{\hbar^2}\frac{(rL-q)^2}{\beta}\right]\right).$$

(d) Making use of the Poisson summation formula,

$$\sum_{r=-\infty}^{\infty} h(r) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi\, h(\phi)\, \mathrm{e}^{2\pi\mathrm{i}n\phi},$$

show that this expression is consistent with that obtained in part (a). *You may note that* $\displaystyle\int_{-\infty}^{\infty} dx\, \exp\left[-\frac{a}{2}x^2 + ibx\right] = \left(\frac{2\pi}{a}\right)^{1/2} \exp\left[-\frac{b^2}{2a}\right]$.

4

(a) Show that the bosonic ($\zeta = 1$), respectively fermionic ($\zeta = -1$), coherent states

$$|\phi\rangle = \exp\left[\zeta \sum_n \psi_n a_n^\dagger\right],$$

are eigenstates of the associated bosonic, respectively fermionic, annihilation operators, a_n, with eigenvalue ψ_n. In both cases, explain the nature of the eigenvalues, ψ_n.

(b) By calculating its commutator with the field operators a_n and $a_n^\dagger$, show that the operator

$$\int d[\bar{\psi}, \psi] e^{-\sum_n \bar{\psi}_n \psi_n} |\psi\rangle\langle\psi|,$$

provides the resolution of identity for bosons or fermions on the Fock space. Explain what is meant by $d[\bar{\psi}, \psi]$ and the domain of integration.

(c) Using this result, show that the quantum partition function for the many-body Hamiltonian, $\hat{H}$, can be expressed as

$$\mathcal{Z} = \int d[\bar{\psi}, \psi] e^{-\sum_n \bar{\psi}_n \psi_n} \langle \zeta\psi | e^{-\beta(\hat{H} - \mu\hat{N})} |\psi\rangle,$$

where $\hat{N}$ denotes the total number operator, μ the chemical potential, and $\beta = 1/k_{\mathrm{B}}T$.

(d) Starting with this result, show that the quantum partition function can be expressed as the field integral,

$$\mathcal{Z} = \int_{\bar{\psi}(\beta)=\zeta\bar{\psi}(0), \psi(\beta)=\zeta\psi(0)} D(\bar{\psi}, \psi) e^{-S[\bar{\psi}, \psi]},$$

where

$$S[\bar{\psi}, \psi] = \int_0^\beta d\tau \left[\sum_n \bar{\psi}_n (\partial_\tau - \mu)\psi_n + H(\bar{\psi}, \psi)\right].$$

(e) For the quantum harmonic oscillator, $\hat{H} = \hbar\omega(a^\dagger a + 1/2)$, explain the relation between the field integral and the Hamiltonian form of the Feynman path integral.

END OF PAPER