UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 9 June, 2015 9:00 am to 12:00 pm

PAPER 80

COMPLEX AND BIOLOGICAL FLUIDS

There are ***THREE*** *questions in total.*

Attempt all ***THREE.***

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A two-dimensional sheet swims in a viscous fluid by deforming its shape as a travelling wave. The material points on the sheet, denoted $[x_s, y_s(x,t)]$ in Cartesian coordinates, are given in the swimming frame by

$$x_s = x, \quad y_s = y_0 \sin(kx - \omega t),$$

where $k > 0$ is the wavenumber, $\omega > 0$ the frequency and $y_0 > 0$ the waving amplitude. A Newtonian fluid of viscosity μ is located in the $y > 0$ domain. The unknown swimming speed of the sheet is denoted $-U\mathbf{e}_x$ and we assume that $y_0 k \ll 1$ ($\mathbf{e}_x$ is the unit vector in the x direction). The sheet is swimming in a Newtonian fluid which is allowed to slip past its surface. Using $\mathbf{u}_s$ to denote the velocity of the material points of the sheet, the appropriate slip boundary condition for the fluid velocity, $\mathbf{u}$, is given, in the limit $y_0 k \ll 1$, by

$$\mathbf{u} - \mathbf{u}_s = \gamma \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \mathbf{e}_x \ \text{ at } \ (x,y) = (x_s, y_s),$$

where $\gamma \geqslant 0$ is a constant with dimensions of length and $\mathbf{u} = (u_x, u_y)$. The no-slip boundary condition corresponds to the particular case $\gamma = 0$.

Assuming a two-dimensional flow and solving for it in the swimming frame, state the equation satisfied by the velocity streamfunction ψ and its boundary conditions. Non-dimensionalise the problem and introduce $\epsilon \equiv y_0 k$ and $\delta \equiv k\gamma$. Solve for the dimensionless streamfunction at order ϵ for all values of δ. Show that the solution at that order is the same as in the no-slip case. Compute the dimensionless swimming speed at order ϵ^2 (you do not need to solve for the entire flow field at order ϵ^2). Deduce that slip ($\delta > 0$) always lead to faster swimming than a no-slip boundary condition.

[*Hint: You can use without proving it that the general travelling-wave solution to* $\nabla^4 f(x,y,t) = 0$ *is*

$$f(x,y,t) = A + By + Cy^2 + Dy^3 + \mathcal{R}\left\{ \sum_n \left[E_n e^{-ny} + F_n e^{ny} + y(G_n e^{-ny} + H_n e^{ny}) \right] e^{in(x-t)} \right\}]$$

2

Locomotion is induced in a Newtonian fluid by the planar deformation of an inextensible flagellum. Give the value of the hydrodynamic force density acting on the flagellum using the framework of resistive-force theory and briefly explain how the resistance coefficients depend on the fluid and geometrical parameters.

The shape of the flagellum is described in Cartesian coordinates by material points $[x, y(x,t)]$ where we assume $|\partial y/\partial x| \ll 1$ everywhere. The flagellum is spatially-periodic with period λ in the x direction and the deformations are periodic in time with period T. Integrating the hydrodynamic forces, calculate the leading order time-averaged propulsive force, F_x, acting on the flagellum over one wavelength in the x direction. Calculate the leading-order time-averaged rate of working, $\dot{W}$, of the flagellum against the fluid over one wavelength.

Consider a functional $J[y] \equiv F_x + \Gamma(\dot{W} - \dot{W}_0)$ where Γ is a constant Lagrange multiplier and $\dot{W}_0$ a constant. By considering a small periodic change in the shape of the flagellum, $y \to y + \delta y$, calculate the resulting change in the functional, $J \to J + \delta J$. Using integration by parts, and enforcing periodicity, derive the equation satisfied by $y(x,t)$ ensuring that $\delta J = 0$ for all values of δy. Deduce that the deformation maximising the propulsive force for a given rate of working is a travelling wave.

3

The fluid velocity induced at location $\mathbf{x}$ by a steady, low-Reynolds number, axisymmetric swimmer located at $\mathbf{x}'$ is well approximated by a stresslet

$$\mathbf{u}(\mathbf{x}) = \mathbf{G}(\mathbf{x} - \mathbf{x}'), \quad \mathbf{G}(\mathbf{r}) = \frac{S}{8\pi\mu}\left[-\frac{\mathbf{r}}{r^3} + \frac{3(\mathbf{e}\cdot\mathbf{r})^2\mathbf{r}}{r^5}\right], \quad r = |\mathbf{r}|,$$

where μ is the dynamic viscosity of the fluid, $\mathbf{e}$ the unit vector along the swimmer (stresslet direction), and S the constant stresslet strength. Explain the physical origin of this flow field and how it can be obtained. Comment on the sign of S and its interpretation.

The centre of the swimmer is assumed to undergo circular motion centered at the origin described in Cartesian coordinates as

$$[x'(t), y'(t), z'(t)] = R\,[\cos\omega t, \sin\omega t, 0]\,,$$

where $R > 0$ is the radius of the circle and $\omega > 0$ the frequency or rotation. Furthermore, the swimmer rotates its shape so that its long axis, $\mathbf{e}(t)$, remains always tangent to the circle. By considering a point far from the circle, $|\mathbf{r}| \gg R$, show explicitly that the flow induced by the rotating swimmer averaged over one period of rotation is also a stresslet, for which you will derive the strength, $\tilde{S}$, and direction, $\tilde{\mathbf{e}}$. Deduce that the time-averaged flow is a pusher if the original is a puller, and vice-versa.

END OF PAPER