UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Wednesday, 3 June, 2015 9:00 am to 12:00 pm

## PAPER 8

## TOPICS IN KINETIC THEORY

*There are* ***THREE*** *questions in total.*

*Attempt all* ***THREE****.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

In this question we will consider the following inhomogeneous Liouville equation

$$\begin{cases} \partial_t f(t,x,v) + \sum_{i=1}^{d} \frac{\partial H}{\partial v_i}(t,x,v)\frac{\partial f}{\partial x_i}(t,x,v) & x, v \in \mathbb{R}^d, t > 0 \\ \qquad - \sum_{i=1}^{d} \frac{\partial H}{\partial x_i}(t,x,v)\frac{\partial f}{\partial v_i}(t,x,v) = h(t,x,v) & \\ f|_{t=0} = f_0. & \end{cases} \quad (1)$$

(a) Write the characteristic equations associated to (1) and give conditions on $H$ under which there exists a unique global solutions to these equations (the conditions need not be optimal).

(b) Prove that for any $d$, if $H \in C^2\left(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}\right)$ satisfies conditions under which a unique global solution to the characteristic equations exists, one has that

$$J(t,x,v) = \det\left(\frac{\partial(S_t(x,v))}{\partial(x,v)}\right) = 1$$

for all $t > 0$, $x, v \in \mathbb{R}^d$, where $S_t(x,v) = (X(t), V(t))$ is the solution to characteristic equations with initial data $(x,v)$.

(c) Show that

$$H(X(t), V(t)) = H(x,v).$$

(d) Consider the case where

$$H_\omega(x,v) = \frac{1}{2}|v|^2 + \frac{\omega^2}{2}|x|^2.$$

Show without solving the characteristic equations that in the case where $h = 0$, if $f_0$ is compactly supported and is continuously differentiable, then $f_t$ is also compactly supported *uniformly in* $t$.

(e) Solve the characteristic equation for $H_\omega$ when $d = 3$ and find an explicit solution to the inhomogeneous Liouville equation in that case.

(f) Under the conditions above, show that if $h$ doesn't depend on $t$ and $f_0, h \in L^p_{x,v}\left(\mathbb{R}^d \times \mathbb{R}^d\right) \cap C^1\left(\mathbb{R}^d \times \mathbb{R}^d\right)$ for $1 < p < \infty$, then $f_t \in L^p_{t,x,v}\left([0,T] \times \mathbb{R}^d \times \mathbb{R}^d\right)$ for any $T > 0$, where $f_t$ is the solution for (1). Show, by finding a concrete example, that it doesn't always hold that $f_t \in L^p\left(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d\right)$ even if $f_0 = 0$.

**2**

In this question we will consider the general linear Boltzmann equation

$$\begin{cases} \partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) = \int_{\mathbb{R}^d} k(t,x,v,v_*)f(t,x,v_*)dv_* - a(t,x,v)f(t,x,v)) \\ f|_{t=0} = f_0. \end{cases}$$

where $x, v \in \mathbb{R}^d$, $t \in (0,T)$ for some $T > 0$, $a \geqslant 0$, $f_0 \in L^2_{x,v}\left(\mathbb{R}^d \times \mathbb{R}^d\right)$ and $k$ is such that

$$C^2 = \int_{\mathbb{R}^d\times\mathbb{R}^d} \sup_{t\in[0,T],x\in\mathbb{R}^d} k^2(t,x,v,v_*)dvdv_* < \infty.$$

We say that $f$ is a weak solution to the general linear Boltzmann equation if $f(t,\cdot,\cdot) \in L^2_{x,v}\left(\mathbb{R}^d\times\mathbb{R}^d\right)$ for all $t \in [0,T)$,

$$\sup_{0\leqslant s\leqslant t} \|f(t,\cdot,\cdot)\|_{L^2_{x,v}} < \infty,$$

for any $t < T$ and

$$f(t,x,v) = e^{-\int_0^t a(\tau,x-v(t-\tau),v)d\tau} f_0(x-vt,v)$$
$$+\int_0^t e^{-\int_s^t a(\tau,x-v(t-\tau),v)d\tau} K(f)(s,x-v(t-s),v)ds$$

in the $L^2$ sense, where

$$K(f)(t,x,v) = \int_{\mathbb{R}^d} k(t,x,v,v_*)f(t,x,v_*)dv_*.$$

Our goal will be to show the existence and uniqueness of weak solutions to our equation.

(a) Show that under the above conditions if $g \in L^2_{x,v}\left(\mathbb{R}^d\times\mathbb{R}^d\right)$ then

$$\|K(g)\|_{L^2_{x,v}} \leqslant C\|g\|_{L^2_{x,v}}$$

and conclude that $K$ is a linear bounded operator from $L^2_{x,v}\left(\mathbb{R}^d\times\mathbb{R}^d\right)$ to itself.

(b) Defining

$$F(f_0,a)(t,x,v) = e^{-\int_0^t a(\tau,x-v(t-\tau),v)d\tau} f_0(x-vt,v)$$

show that

$$\sup_{0\leqslant s\leqslant t} \|F(f_0,a)(s,\cdot,\cdot)\|_{L^2_{x,v}} \leqslant \|f_0\|_{L^2_{x,v}}.$$

(c) Defining

$$\tau(f)(t,x,v) = \int_0^t e^{-\int_s^t a(\tau,x-v(t-\tau),v)d\tau} K(f)(s,x-v(t-s),v)ds$$

show that

$$\|\tau(f)(t,\cdot,\cdot)\|_{L^2_{x,v}} \leqslant C \sup_{0\leqslant s\leqslant t} \|f(s,\cdot,\cdot)\|_{L^2_{x,v}}\, t.$$

(d) Show that

$$\|\tau^n(f)(t,\cdot,\cdot)\|_{L^2_{x,v}} \leqslant \frac{C^n t^n}{\sqrt{1\cdot 3\ldots(2n-1)}} \sup_{0\leqslant s\leqslant t} \|f(s,\cdot,\cdot)\|_{L^2_{x,v}}.$$

(e) Prove that the equation

$$(I-\tau)f = F(f_0, a)$$

has a solution that satisfies

$$\sup_{0\leqslant t\leqslant T} \|f(t,\cdot,\cdot)\|_{L^2_{x,v}} \leqslant C_T \|f_0\|_{L^2_{x,v}}$$

for some constant $C_T > 0$ and conclude the existence of weak solution to the linear Boltzmann equation.

(f) Show that under our conditions, the weak solution to our equation is unique.

**3**

In this question we will investigate another physically important equation, the Fokker Planck equation, and the problem of convergence to equilibrium in it. The simple Fokker Planck equation on $\mathbb{R}^d$ is given by

$$\begin{cases} \partial_t f(t,v) = \Delta f(t,v) + \nabla \cdot (f(t,v)v) \\ f|_{t=0} = f_0. \end{cases} \tag{1}$$

(a) Show that $\gamma(v) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|v|^2}{2}}$ is a stationary solution to (1).

(b) Denoting by

$$M(t) = \int_{\mathbb{R}^d} f(t,v)dv,$$

$$u(t) = \frac{1}{M(t)} \int_{\mathbb{R}^d} v f(t,v)dv,$$

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^d} |v|^2 f(t,v)dv.$$

Assuming enough regularity and decay at infinity, find differential equations to $M(t)$, $\{u_i(t)\}_{i=1,\dots,d}$, $E(t)$ and solve them. Conclude that

$$M(t) = M(0)$$

$$u(t) \underset{t\to\infty}{\longrightarrow} 0$$

and

$$E(t) \underset{t\to\infty}{\longrightarrow} \frac{dM(0)}{2}.$$

From this point onward we will assume that $f \geqslant 0$ for all $t$, $d = 1$ and $M(0) = \int_{\mathbb{R}} f_0(v)dv = 1$. As in this case the mass of $f$ and $\gamma$ are equal, and in the limit as time goes to infinity so are the momentum and energy, it is conceivable to conjecture that $f$ will converge to $\gamma$. Finding the right 'distance' to measure this convergence will occupy us for the rest of the problem. You may assume that $f$ is regular enough and decays at infinity sufficiently fast from this point onward.

(c) Define the relative entropy of $f$ with respect to $\gamma$ as

$$H(f|\gamma) = \int_{\mathbb{R}} f \log\left(\frac{f}{\gamma}\right) dv = \int_{\mathbb{R}} \frac{f}{\gamma} \log\left(\frac{f}{\gamma}\right) \gamma dv$$

. Using the fact that $x \log x - x + 1 \geqslant 0$ for all $x \geqslant 0$ show that $H(f|\gamma) \geqslant 0$.

(d) Show the following equalities

$$\int_{\mathbb{R}} f'' \log f dv = -\int_{\mathbb{R}} \frac{|f'|^2}{f} dv,$$

$$\int_{\mathbb{R}} (fv)' \log f dv = 1,$$

and conclude that if $f$ solves (1) then

$$\frac{d}{dt}H(f_t|\gamma) = -\left(\int_{\mathbb{R}} \frac{|f_t'|^2}{f_t}dv + \int_{\mathbb{R}} v^2 f_t dv - 2\right) = -\int_{\mathbb{R}} \left|\frac{d}{dv}\log\left(\frac{f_t}{\gamma}\right)\right|^2 f_t dv = -I(f_t|\gamma),$$

which proves that $H(f|\gamma)$ is decreasing in time. $I(f|\gamma)$ is called the relative Fisher Information of $f$ with respect to $\gamma$.

(e) One can show that if $f$ solves (1) then

$$I(f_t|\gamma) \leqslant -\frac{1}{2}\frac{d}{dt}I(f_t|\gamma).$$

Use the above to show that $I(f_t|\gamma)$ converges to zero exponentially fast, and together with with the monotonicity and positivity of $H$ show that $H(f_t|\gamma)\frac{d}{dt}H(f_t|\gamma) \in L^1_t(\mathbb{R}^+)$.

(f) The above implies that $\lim_{t\to\infty} H(f_t|\gamma) = 0$ (you don't need to prove it). Use the fundamental theorem of calculus to conclude that for any $t > 0$

$$H(f_t|\gamma) \leqslant -\frac{1}{2}\frac{d}{dt}H(f_t|\gamma),$$

and give an explicit estimation to the convergence to equilibrium of $f_t$ using $H$ and the initial data alone..

## END OF PAPER