UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2015 1:30 pm to 4:30 pm

PAPER 79

FLUID DYNAMICS OF CLIMATE

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The linearised shallow water equations for a single layer of fluid of constant depth H on a $f-$plane are

$$\begin{aligned}\mathbf{u_t} + \mathbf{f} \times \mathbf{u} &= -g\nabla_h \eta, \\ \eta_t + H\nabla_h \cdot \mathbf{u} &= 0,\end{aligned}$$

where a subscript implies a partial derivative, $\mathbf{u}$ is the horizontal velocity and η is the free surface elevation.

Show that these equations reduce to a single equation for η

$$\eta_{tt} + f^2\eta - c^2(\eta_{xx} + \eta_{yy}) = -H\mathbf{f} \cdot \mathbf{q}, \qquad (1)$$

where $\mathbf{q} = \nabla_h \times \mathbf{u} - \eta\mathbf{f}/H$, $c^2 = gH$ and $f = |\mathbf{f}|$.

Consider a layer of fluid for which at $t = 0$, $\eta = 0$ everywhere and the region $|y| \leqslant a$ is set into motion with uniform velocity U along the $x-$axis, while the fluid outside this region is at rest. Show that this initial state adjusts to a final state given by

$$\frac{\eta}{H} = \frac{U}{c}\begin{cases} -e^{-a/\lambda}\sinh(y/\lambda), & |y| \leqslant a, \\ -e^{-y/\lambda}\sinh(a/\lambda), & y > a, \\ e^{y/\lambda}\sinh(a/\lambda), & y < -a, \end{cases}$$

where λ is a length scale that should be determined. Evaluate the resulting velocity field and sketch the final free surface elevation and velocity profile for the cases $a/\lambda \ll 1$ and $a/\lambda \gg 1$.

Starting from (1) or otherwise show that the energy E_f of the final state is given by

$$E_f = -\frac{gH}{2f}\int \eta|\mathbf{q}|dA,$$

where the integral is taken over the whole domain. Hence show that the ratio of E_f to the energy E_i of the initial state is

$$\frac{E_f}{E_i} = \frac{1 - e^{-2a/\lambda}}{2a/\lambda}$$

and comment on the cases $a/\lambda \ll 1$ and $a/\lambda \gg 1$.

2

Explain what is meant by the 'beta-plane' approximation for motion on a rotating sphere.

Consider a layer of fluid of constant depth H in a frame of reference rotating with angular velocity $\frac{1}{2}f\hat{\mathbf{z}}$. The free surface height is given by $z = \eta(x, y, t)$. For motion with horizontal scales $L \gg H$ and started from rest show that the horizontal velocities (u, v) are independent of depth. Starting from the linearised equations of motion under these approximations show that the 'transport velocities' $U \equiv Hu, V \equiv Hv$ satisfy

$$\begin{aligned} U_t - fV &= -c^2\eta_x, \\ V_t + fU &= -c^2\eta_y, \\ \eta_t + U_x + V_y &= 0, \end{aligned}$$

where a subscript implies a partial derivative and where $c = \sqrt{gH}$.

For motion on a beta plane with $f = f_0 + \beta y$ show that these equations reduce to

$$\mathcal{F}\left[\eta_{xx} + \eta_{yy} - \frac{1}{c^2}\mathcal{F}\eta\right]_t = \beta(\eta_{xtt} + 2f_0\eta_{yt} - f_0^2\eta_x),$$

where $\mathcal{F} \equiv \frac{\partial^2}{\partial t^2} + f_0^2$.

Show that low frequency waves $\omega \ll f$ satisfy

$$\left[\nabla^2\eta - a^2\eta\right]_t = \beta\eta_x, \tag{1}$$

where $a = \frac{f_0}{c}$. Derive and sketch the dispersion relation.

Show that for waves with frequency ω the wave vector (k, l), where $k > 0$, lies on a circle in the $k - l$ plane with centre $(-\beta/2\omega, 0)$, and radius $\sqrt{\beta^2/4\omega^2 - a^2}$.

Consider a wave in $x > 0$ with wavenumber $\mathbf{k_i} = (k_i, l_i)$ and frequency ω_i satisfying (1) incident on a rigid boundary at $x = 0$. Show that the reflected wave also has frequency ω_i and wavenumber (k_r, l_i), has the same amplitude as the incident wave, and determine k_r.

3

Consider the following forced two-layer quasi-geostrophic equations:

$$\frac{Dq_1}{Dt} = W,$$
$$\frac{Dq_2}{Dt} = 0,$$

where D/Dt is the material derivative and W is the wind stress curl. The potential vorticity in each layer can be written

$$q_1 = f + \nabla_h^2 \psi_1 - k_I^2(\psi_1 - \psi_2),$$
$$q_2 = f + \nabla_h^2 \psi_2 - k_I^2(\psi_2 - \psi_1),$$

where ∇_h^2 is the horizontal Laplacian operator, and k_I is the constant internal deformation wavenumber associated with equal layer depths in a state of rest.

(i) Derive an expression for Sverdrup balance, stating all necessary assumptions. Qualitatively discuss how the flow would respond to a negative wind stress curl, and give a physical interpretation for each term in the potential vorticity conservation equation.

(ii) For a uniform wind stress curl, analyse the stability of the Sverdrup balance from part (i) to small amplitude perturbations of the form:

$$\psi_n' = \widehat{\psi}_n e^{i(kx+ly-\sigma t)},$$

where $n = 1, 2$. Derive an equation relating σ, k, and l. *Note, you don't need to obtain an explicit expression for* σ.

(iii) Derive a necessary condition for instability for the perturbations described in part (ii), with $k = 0$. Discuss the influence of $\beta \equiv df/dy$, the northward gradient of the Coriolis parameter, on growing perturbations.

4

The stratified quasi-geostrophic equations on an f-plane can be written

$$\frac{Dq}{Dt} = 0,$$

where

$$q = f_0 + \nabla_h^2 \psi + \frac{f_0^2}{N^2}\frac{\partial^2 \psi}{\partial z^2},$$

D/Dt is the material derivative, ψ is the streamfunction associated with the horizontal velocity, and the buoyancy frequency N is the buoyancy frequency, assumed to be constant.

(i) Consider a flow $\overline{\mathbf{u}}(y, z)$ in thermal wind balance with the buoyancy

$$\overline{b}(y, z) = A(y) + B(z),$$

where $A(y)$ and $B(z)$ are arbitrary functions and $\overline{\mathbf{u}} = 0$ at $z = 0$. Derive an expression for $\overline{\mathbf{u}}$ and the potential vorticity $\overline{q}$ associated with this flow in terms of A, B, f_0, and N.

(ii) A semi-infinite fluid is bounded from below by a flat, rigid surface at $z = 0$, with uniform buoyancy gradients in both directions, $\overline{b} = M^2 y + N^2 z$, where M and N are constants. Consider linear perturbations to a state of thermal wind balance of the form

$$\psi' = \widehat{\psi}(z) e^{i(kx+ly-kct)}. \qquad (1)$$

Deduce the functional form of $\widehat{\psi}(z)$. You may assume that perturbations decay as $z \to \infty$.

(iii) Derive a dispersion relation for waves of the form given in (1), and find the height at which the phase speed exactly opposes the thermal wind.

(iv) Does this system described permit unstable linear perturbations? Comment on the difference between this system and the same flow bounded from above and below by rigid horizontal surfaces.

END OF PAPER