UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2015 9:00 am to 11:00 am

PAPER 78

FLUID DYNAMICS OF THE SOLID EARTH

There are ***THREE*** *questions in total.*

You may attempt any number of questions.

Full marks can be gained from complete answers to ***TWO*** *questions.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Starting from Darcy's law, show that the pressure field in a uniform porous medium satisfies Laplace's equation.

An infinite porous medium of uniform permeability Π is saturated with water of density ρ, dynamic viscosity μ and temperature $T_\infty < T_m$, where T_m is the equilibrium freezing temperature at the initial pressure $p = 0$. The equilibrium freezing temperature T_e at pressure p is given by

$$\rho_s L \frac{T_m - T_e}{T_m} = p\left(1 - \frac{\rho_s}{\rho}\right),$$

where $\rho_s < \rho$ is the density of ice and L is the specific latent heat of fusion of water into ice.

A spherical region with radius $a(t)$ of ice-filled pores forms within the porous medium. Use mass conservation to determine the radial Darcy velocity $u = u(r,t)$. In particular, show that

$$\phi(\rho - \rho_s)\dot{a} = \rho u(a,t),$$

where ϕ is the porosity of the medium. Hence, or otherwise, determine the pressure field $p(r,t)$.

Write down a complete set of equations and boundary conditions to describe the spherically-symmetric evolution of this system, ignoring any effects of surface energy. Given that $L/c_p(T_m - T_\infty) \gg 1$, where c_p is the specific heat capacity of water, ice and the matrix of the porous medium, explain carefully, with a scaling analysis of the governing equations, why the temperature field can be treated as quasi-steady, satisfying Laplace's equation. Find a solution for the temperature field T and the radius of the frozen region $a(t)$ in this regime.

Determine and comment on the behaviour of $p(a)$, $T(a)$ and $a(t)$ as the permeability decreases.

2

When assessed on geological time scales the building of mountain ranges at convergent plate margins may be considered to be approximately viscous. Consider a simplified, two-dimensional geometry in which a rigid tectonic plate converges with a pre-existing mountain range of density ρ and viscosity μ at velocity U. If the converging plate remains undeformed (ie. is horizontal at all times) determine the predominantly horizontal *fluid* velocity of the mountain range in the limit where gravitational spreading of the mountains is balanced by viscous stresses exerted by the underthrusting plate. Integrate vertically to find the horizontal mass flux and determine the topography of the mountain range, $h(x)$, as a function of the distance from the contact line, x.

In the same geometry, now consider the addition of a pre-existing sedimentary basin of equal density ρ and viscosity $\mu_s < \mu$ that covers the plate to a depth $d(x)$, thereby reducing the effective viscous coupling between the mountain range and the plate. By considering the *fluid* velocities within the mountain and sedimentary layer, derive the set of coupled equations governing the topography $h(x)$ of the mountain range (as measured from the plate) and the thickness of the basal sedimentary layer $d(x)$, assuming that the sedimentary layer has thickness d_∞ far from the contact line. Show that, in steady state, there is a relationship between the topography of the mountain and the sedimentary layer thickness given by

$$(h-d)^2 + \frac{3d}{2M}\left(2 - \frac{d}{d-d_\infty}\right)(h-d) + \frac{d^2}{M}\left(\frac{3}{2} - \frac{d}{d-d_\infty}\right) = 0,$$

where $M \equiv \mu_s/\mu$ is the ratio of sedimentary basin and mountain viscosities.

Evaluate the height of the contact line above the plate, and the slope of the mountain range as a function of M. In addition, find the analytical expression for the profile of the sedimentary basin in front of the contact line.

3

A collision of asteroidal bodies results in a large, nearly pure iron asteroid of radius R with a uniform interior temperature $\overline{T} > T_m$ above the melting point of pure iron. Radiative cooling at the surface maintains a surface temperature $T_s < T_m$, and hence drives the formation of an iron crust of thickness $a(t)$ and a cold thermal boundary layer. A combination of two potential mechanisms thus controls the asteroid's evolution; cooling through thermal convection and the viscous deformation and sinking of the solidified crust to form the asteroidal core, which can be treated as a convection process with the crust acting as a dense, viscous thermal boundary layer.

First consider the diffusive growth of the crust and thermal boundary layer while $a \ll R$. In your analysis of the diffusive growth you may neglect differences in specific heat c_p and thermal conductivity k between phases in your analysis of boundary layer growth. You may also neglect density differences between the phases except insofar as they affect the buoyancy. Find the rate of crustal growth as a function of latent heat per unit mass L, the cooling from the roof $\Delta T = T_m - T_s$, and the interior temperature $\overline{T}$. You should determine the similarity solution for the general problem and then find an approximate solution in the limit of large Stefan number, $S = L/c_p\Delta T$. This approximation may be used for the remainder of the question.

The cold thermal boundary layer in the liquid and the viscous solid crust are both prone to convective instabilities. Estimate the ratio of characteristic timescales of the thermal boundary layer and viscous crust, assuming a driving temperature difference $\Delta T = 125$ K, latent heat $L = 2.7 \times 10^5$ J kg^{-1}, specific heat $c_p = 850$ J kg^{-1}K^{-1}, ratio of viscosities $\mu_s/\mu_l \simeq 10^{18}$, thermal coefficient of expansion of the liquid $\alpha = 10^{-4}$ K^{-1}, densities $\rho_s = 8500$ kg m^{-3} and $\rho_f = 8000$ kg m^{-3}, and for a range of interior temperatures $\overline{T} - T_m = (10^{-2} - 10^2)(T_m - T_s)$. Using scaling arguments, or otherwise, construct expressions for the thermal flux and solid flux averaged over many overturn cycles.

At late times, the remaining liquid interior is at the melting temperature $\overline{T} = T_m$. The viscous instability of the crust releases dense, solid iron which sinks to form an inner core. Estimate the size of the inner core radius $b(t)$ as a function of the radius of the asteroid R, using the expression for the solid flux derived earlier.

END OF PAPER