UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Wednesday, 3 June, 2015 9:00 am to 11:00 am

## PAPER 77

## SOUND GENERATION AND PROPAGATION

*Attempt no more than* ***TWO*** *questions.*

*There are* ***THREE*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

**(a)** Lighthill's equation describing aerodynamic sound generation is

$$\frac{\partial^2 \rho'}{\partial t^2} - {c_0}^2 \nabla^2 \rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}, \qquad (\dagger)$$

where $T_{ij} = \rho u_i u_j + (p' - {c_0}^2 \rho')\delta_{ij} - \sigma_{ij}$ is a quadrupole distribution.

**(i)** Show that the far-field sound generated by a compact quadrupole distribution is

$$\rho'(\boldsymbol{x}, t) = \frac{x_i x_j \ddot{S}_{ij}(t - |\boldsymbol{x}|/c_0)}{4\pi {c_0}^4 |\boldsymbol{x}|^3} \qquad \text{where} \qquad S_{ij}(\tau) = \int T_{ij}(\boldsymbol{y}, \tau)\, \mathrm{d}^3 y$$

and $\dot{}$ denotes differentiation with respect to $t$. Show further that $\rho'$ scales like $O(m^4)$, where $m$ is the fluctuating Mach number.

**(ii)** Now consider motion at a single frequency $\omega$. Show that in two dimensions $\rho'$ scales like $O\big(m^{7/2}\big)$.

*[In two dimensions, the free-space Green's function for the Helmholtz equation has the far-field form*

$$\widetilde{G}(\boldsymbol{x};\, k_0) \sim \frac{\exp\{-\mathrm{i}k_0|\boldsymbol{x}|\}}{\sqrt{k_0|\boldsymbol{x}|}} \qquad \textit{as} \qquad |\boldsymbol{x}| \to \infty$$

*where $k_0 = \omega/c_0$.]*

**(b)** Consider a shock wave (i.e. a surface of discontinuity) in a fluid, with equation $S(\boldsymbol{x}, t) = 0$, with $S > 0$ on one side of the shock and $S < 0$ on the other side.

**(i)** Suppose

$$\frac{\partial a^+}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{b}^+ = 0 \qquad \text{for} \qquad S(\boldsymbol{x}, t) > 0,$$

$$\frac{\partial a^-}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{b}^- = 0 \qquad \text{for} \qquad S(\boldsymbol{x}, t) < 0,$$

with $a^\pm$ and $\boldsymbol{b}^\pm$ continuously differentiable functions. Show that

$$\frac{\partial a}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{b} = \boldsymbol{f} \cdot \boldsymbol{n} |\boldsymbol{\nabla} S| \delta(S),$$

where $a = a^+ H(S) + a^- H(-S)$, $\boldsymbol{b} = \boldsymbol{b}^+ H(S) + \boldsymbol{b}^- H(-S)$, $\boldsymbol{n}$ is a unit vector normal to the shock, and $\boldsymbol{f}$ is an expression that should be found explicitly. *[Note that $\boldsymbol{f}$ involves $\boldsymbol{v}$, the velocity of the shock.]*

**(ii)** Hence, find the equivalent of Lighthill's equation ($\dagger$) for $\rho' = \rho^+ H(S) + \rho^- H(-S) - \rho_0$. Give a physical interpretation of each of the source terms in your equation. Without further calculation, explain whether you would still expect $\rho'$ to scale like $O(m^4)$ in the far-field compact limit with a shock present.

**2**

*This question considers 2D harmonic motion, so there is no $z$ dependence and time dependence $e^{i\omega t}$ is assumed. You may like to consider* $\mathrm{Im}(\omega) < 0$.

**(a)** An elastic membrane is stretched along $y = 0$ with tension $T$. The membrane undergoes small oscillations, with its displacement $\eta(x)$ in the $y$-direction governed by

$$-m\omega^2\eta = T\frac{\partial^2\eta}{\partial x^2} - \Big[p\Big]_{0-}^{0+},$$

where $[p]_{0-}^{0+}$ is the pressure difference across the membrane. Above and below the membrane is a compressible fluid, and motion of the membrane causes waves in the fluid with unsteady pressure $p$ satisfying Helmholtz equation $\nabla^2 p + k_0^2 p = 0$, where $k_0 = \omega/c_0$. The boundary condition of no flow through the membrane implies

$$\eta = \frac{1}{\rho_0\omega^2}\left.\frac{\partial p}{\partial y}\right|_{y=0-} = \frac{1}{\rho_0\omega^2}\left.\frac{\partial p}{\partial y}\right|_{y=0+}.$$

By Fourier transforming in the $x$ direction, find the dispersion relation $D(\omega, k) = 0$ for waves on the membrane.

**(b)** Now consider a semi-infinite elastic membrane pinned at $(x, y) = (0, 0)$ and stretched along $y = 0$ for $x < 0$. A wave on the membrane is incident from $x = -\infty$, giving a displacement $\eta_I = e^{-ik_I x}$ where $D(\omega, k_I) = 0$ and $\mathrm{Im}(k_I) < 0$ (thus corresponding to a right-propagating wave). Because the membrane is only semi-infinite, the incoming wave scatters into sound in the fluid and left-propagating waves on the membrane.

Let $\eta = \eta_I + \eta'$ and $p = p_I + p'$, where $p_I$ is the pressure corresponding to displacement $\eta_I$ from your solution to (a). The governing equations are then

$$\begin{aligned}
0 &= \nabla^2 p' + k_0^2 p' && \forall x,\\
\eta' &= \frac{1}{\rho_0\omega^2}\left.\frac{\partial p'}{\partial y}\right|_{y=0-} = \frac{1}{\rho_0\omega^2}\left.\frac{\partial p'}{\partial y}\right|_{y=0+} && \forall x,\\
\Big[p'\Big]_{0-}^{0+} &= m\omega^2\eta' + T\frac{\partial^2\eta'}{\partial x^2} && x < 0,\\
\Big[p'\Big]_{0-}^{0+} &= -\Big[p_I\Big]_{0-}^{0+} && x > 0.
\end{aligned}$$

Briefly justify why these are the governing equations. By taking full- and half-range Fourier transforms in the $x$ direction, find the Wiener–Hopf equation

$$K(k)P^-(k) + \eta^+(k) = \frac{T}{m\omega^2 - Tk^2}\left.\frac{\partial\eta}{\partial x}\right|_{x=0} + \frac{i\gamma(k)}{(k - k_I)\gamma(k_I)},$$

where $\gamma^2 = k^2 - k_0^2$ with $\mathrm{Re}(\gamma) > 0$ for real $k$, $P^-$ is the left half-range Fourier transform of $p'(x, 0+)$, $\eta^+$ is the right half-range Fourier transform of $\eta'$, and $K(k)$ is the Wiener–Hopf kernel to be given explicitly.

Assuming that $K(k)$ can be factorized as $K(k) = K^+(k)K^-(k)$ (which should not be found explicitly), what singularities do $K^+$ and $K^-$ have? Describe the zeros of $K^+$ and $K^-$ (which you need not find explicitly). Assuming the existence of any factorizations you need (which should not be found explicitly), and that the resulting entire function $E(k)$ is identically zero, give an integral expression for the pressure $p$ in the fluid. Show also that $\eta^-$ has a pole in the lower half plane. What does this pole represent physically, and how might it imply a value for the as yet undetermined constant $\partial\eta/\partial x|_{x=0}$.

**3**

Burgers' equation is

$$\frac{\partial f}{\partial Z} - f\frac{\partial f}{\partial \theta} = \alpha\frac{\partial^2 f}{\partial \theta^2}.$$

The inviscid Burgers' equation is obtained by setting $\alpha = 0$.

**(a)** Show that the inviscid Burgers' equation with initial conditions $f(0,\theta) = f_0(\theta)$ has solution $f\big(Z, \theta_0 - f_0(\theta_0)Z\big) = f_0(\theta_0)$. Show also that if there is a weak shock at $\theta_s(Z)$ then

$$\frac{\mathrm{d}\theta_s}{\mathrm{d}Z} = -\tfrac{1}{2}\lim_{\delta\to 0}\Big(f(Z,\theta_s+\delta) + f(Z,\theta_s-\delta)\Big).$$

Solve the inviscid Burgers' equation for the initial conditions

$$f(0,\theta) = \begin{cases} 0 & \theta < 0 \\ U & \theta > 0 \end{cases} \qquad (*)$$

being careful to distinguish between $U < 0$ and $U > 0$.

*[It may help to sketch the characteristics first. For $U < 0$, think of $f(0,\theta)$ as being continuous but very steep at $\theta = 0$.]*

**(b)** For $\alpha \neq 0$, show that the Cole–Hopf transformation

$$f = 2\alpha\frac{\partial}{\partial\theta}\log\psi$$

can be used to solve Burgers' equation when $\psi$ satisfies a diffusion equation. Given that the general solution to the diffusion equation is

$$\psi(Z,\theta) = \frac{1}{\sqrt{4\pi\alpha Z}}\int_{-\infty}^{\infty}\psi(0,\phi)\exp\left\{-\frac{(\phi-\theta)^2}{4\alpha Z}\right\}\mathrm{d}\phi,$$

show that the solution to the full Burgers' equation for the initial conditions given in (*) is

$$f(Z,\theta) = \frac{U}{1 + J(Z,\theta)\exp\big\{-U(2\theta+UZ)/4\alpha\big\}},$$

where

$$J(Z,\theta) = \frac{\displaystyle\int_{\theta}^{\infty}\exp\big\{-y^2/4\alpha Z\big\}\,\mathrm{d}y}{\displaystyle\int_{-(\theta+UZ)}^{\infty}\exp\big\{-y^2/4\alpha Z\big\}\,\mathrm{d}y}.$$

What happens (i) as $\theta\to\infty$, (ii) as $\theta\to-\infty$, and (iii) when $J = 1$? How does this compare with your inviscid solution found in (a)?

$$\left[\textit{Note:}\qquad \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty}e^{-t^2}\,\mathrm{d}t \quad\sim\quad \begin{cases}\dfrac{e^{-x^2}}{x\sqrt{\pi}} & \textit{as}\quad x\to+\infty\\[2ex] 2 & \textit{as}\quad x\to-\infty\end{cases}.\right]$$

**END OF PAPER**