UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2015 1:30 pm to 3:30 pm

PAPER 76

DIRECT AND INVERSE SCATTERING OF WAVES

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider an acoustic field generated by a time-harmonic wave $\psi(\mathbf{r})$ with wavenumber k propagating in a 2-dimensional medium with refractive index $n(\mathbf{r})$.

Assume that wave propagation is at small angles to the horizontal direction x. Derive the parabolic equation for the propagation of the reduced wave $E = \psi e^{-ikx}$ in the refractive medium, and state under what conditions it is valid.

(b) Consider now the case where the acoustic wave propagating mostly in the horizontal direction is an incident Gaussian beam

$$E(0, z) = e^{-\frac{z^2}{D^2} - i\frac{kz^2}{2F}} \tag{1}$$

propagating in free space. Derive an expression for this beam $E(x, z)$ in free space at a propagation distance x.

[You may wish to use the following formula for a Gaussian integral:

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}az^2+bz} dz = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}$$

valid for $\mathrm{Re}[a] > 0$.*]*

(c) Now assume that the propagated beam $E(x, z)$ encounters an extended randomly refractive medium at $x = x_0$, so that the refractive index $n(\mathbf{r}) = n(x, z)$ for $x > x_0$ is given by $n(\mathbf{r}) = 1 + \mu W(\mathbf{r})$, with $\mu \ll 1$, where W is normally distributed and statistically stationary, with mean and variance given by $< W >= 0$ and $< W^2 >= 1$.

Consider the first moment of the field generated by this wave
$m_1 =< E(x, z) >$.

Is m_1 independent of the transverse direction z? Explain.

(d) Derive an equation for the propagation of the first moment of the field generated by the Gaussian beam in the extended random medium, for $x > x_0$.

2

Consider the 3-dimensional scattering problem of a time-harmonic plane wave $\psi_i(\mathbf{r}) = e^{ik\hat{\mathbf{r}}_0 \cdot \mathbf{r}}$ incident from the direction $\hat{\mathbf{r}}_0$ in free space, being scattered by an extended inhomogeneity with refractive index $n(\mathbf{r})$ occupying a finite volume V.

(a) Derive the Born approximation for the scattered field $\psi_s(\mathbf{r})$.

(b) Use the first Born approximation for the scattered field and its far field behaviour to calculate the time-averaged energy flux in the direction $\hat{\mathbf{r}}$, E_s, at a distance R in the far field of the inhomogeneity, in terms of the far field pattern $f_\infty(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0)$.

Calculate also the time-averaged energy flux E_i (in the direction $\hat{\mathbf{r}}_0$) of the incident wave.

(c) The differential scattering cross section $\sigma_d(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0)$ in the direction $\hat{\mathbf{r}}$, caused by an incident field in the direction $\hat{\mathbf{r}}_0$, is defined as:

$$\sigma_d(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0) = \lim_{R\to\infty} \left(\frac{R^2 E_s}{E_i} \right) . \tag{1}$$

Calculate $\sigma_d(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0)$ in the first Born approximation, using the quantities derived in (b), to obtain an expression in terms of the far field pattern $f_\infty(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0)$.

(d) Now assume that the inhomegeneity in V has randomly varying refractive index $n(\mathbf{r}) = 1 + \mu W(\mathbf{r})$, with $\mu \ll 1$, where W is normally distributed and statistically stationary, with mean and variance given by $< W >= 0$ and $< W^2 >= 1$. Also assume that the size of the volume V is much greater that the correlation distance of the refractive index.

Using the the differential scattering cross section in terms of the far field pattern $f_\infty(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0)$ obtained in (c), derive an expression for the ensemble average of $\sigma_d(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0)$ in terms of the power spectrum of the inhomogeneity $S(k_s)$, where k_s is related to the difference between $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}_0$, neglecting any terms higher than second order in μ.

3

Consider the equation $Ax = y$, where A is a compact linear operator between two Hilbert spaces: $A : X \mapsto Y$, and $x \in X,\ y \in Y$.

(a) Explain why, when considering the inverse problem of finding x, given A and data y (and especially in the case of noisy data), it is appropriate to seek the solution x of the normal equation

$$A^*Ax = A^*y \ , \tag{1}$$

where A^* denotes the adjoint of A, instead of the solution of $Ax = y$.

(b) Using a fixed-point equation for (1), derive Landweber iteration for the calculation of successive approximations x_n to its solution, and obtain a closed-form expression for the n^{th} iterate, given the choice $x_0 = 0$ for the first iterate.

(c) Landweber iteration can be viewed as a regularisation method, with regularisation parameter $\alpha = 1/n$.

Assume that there exists a singular value system $\{\sigma_i; u_i, v_i\}$ for A. Given noisy data $y^{(\delta)}$, with $\| y^{(\delta)} - y \| \leqslant \delta$, find the filter function $g_\alpha(\sigma_i)$ in the expression for the corresponding solution $x_\alpha^{(\delta)}$ in terms of the singular value system:

$$x_\alpha^{(\delta)} = \sum_{i=1}^{\infty} g_\alpha(\sigma_i)(y^{(\delta)}, u_i)v_i \ . \tag{2}$$

(d) State a condition under which Landweber iteration is convergent, and check that, under this condition, the limit of the filter function as $\alpha \to 0$ is correct, thus showing that this iteration is a regularisation.

END OF PAPER