UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2015 1:30 pm to 4:30 pm

PAPER 75

BIOLOGICAL PHYSICS

*There are **THREE** questions in total.*

*Attempt all **THREE**.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider an infinite cylinder of radius a and negative linear charge density $-\lambda$ immersed in an electrolyte solution with an axisymmetric concentration $n(r)$ of positive charges q and a dielectric constant ϵ. Write down the Poisson-Boltzmann equation relating the charge density $\rho(r) = qn(r)$ to the electrostatic potential $\phi(r)$, introducing an appropriate normalization factor n_0 for the concentration. By further introducing the change of coordinates $u = \ln(r/a)$ and $\Phi = \beta q\phi - 2u$, where $\beta = 1/k_BT$, show that the Poisson-Boltzmann equation takes the form

$$\frac{d^2\Phi}{du^2} = -\frac{4\pi\beta n_0 q^2 a^2}{\epsilon} \mathrm{e}^{-\Phi} \; .$$

Solve this for Φ, assuming $\Phi(a) = 0$ and that $\Phi \to \infty$ and $d\Phi/du \to 0$ as $u \to \infty$. Express the electric field magnitude $E(r)$ for $r \geqslant a$ in terms of λ and ρ and thereby find an expression for the total screening charge density $Q_{\text{screening}}$ around the cylinder in terms of $\rho(r)$. Using the relationship between $d\phi/dr$ at $r = a$ and λ, solve for the normalization constant n_0. Show that "counterion condensation" occurs, such that $n_0 = 0$ for $\lambda < \lambda_c$ and $n_0 > 0$ for $\lambda > \lambda_c$, where $\lambda_c = q/\ell_B$, with ℓ_B the Bjerrum length. Show that $Q_{\text{screening}} = \lambda - \lambda_c$ for $\lambda > \lambda_c$.

2

Suppose an elastic filament of length L and bending modulus A is held parallel to the x-axis, and *clamped* at its left end so that its (presumed small) displacement $h(x)$ from the x-axis satisfies $h(0) = h_x(0) = 0$. If its right-hand end is unconstrained, state the boundary conditions on h that hold there. Using the form of the Euler-Lagrange equation for the appropriate functional for the bending energy, show that the shape of any function $h(x)$ satisfying the above boundary conditions is of the general biharmonic form,

$$W_n(x) = B \sin k_n x + D \cos k_n x + E \sinh k_n x + F \cosh k_n x \; ,$$

where k_n is the nth root of the transcendental equation $\cos kL \cosh kL = -1$. Graphically or otherwise estimate the first positive root k_1. From these results, show that

$$\begin{aligned} W_n(x) = N[&(\sin q_n + \sinh q_n)(\cos k_n x - \cosh k_n x) \\ &- (\cos q_n + \cosh q_n)(\sin k_n x - \sinh k_n x)] \; , \end{aligned} \tag{1}$$

where N is a normalization constant and $q_n = k_n L$. Calculate the mean squared fluctuation of the free end at temperature T. [You may use without proof the relationship $W_n(L)^2 = 4I_n$, where $I_n = (1/L)\int_0^L dx W_n(x)^2$.]

3

In the simplest model of so-called 'phase oscillators', the dynamics of an angular variable $\Delta(t)$ takes the form

$$\dot{\theta} = \omega - \epsilon \sin\theta + \xi(t) \ , \tag{1}$$

where $\omega > 0$ is an intrinsic frequency difference, $\epsilon > 0$ is a coupling constant, and the Langevin noise is Gaussianly distributed with $\langle \xi \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2T_{\rm eff}\delta(t-t')$, with $T_{\rm eff}$ an effective temperature.

(a) Examining first the noise-free case, show by graphing the right-hand-side of (1) that there is a pair of fixed points for $\theta \in (0, 2\pi)$ for $\omega < \epsilon$ and no fixed points for $\omega > \epsilon$. In the former case, find the fixed points analytically and determine the stability of both. In this type of overdamped dynamics, the right-hand-side of the equation can be interpreted as an effective force. Find the potential $V(\theta)$ which governs that force. Explain graphically or otherwise the distinction, in terms of $V(\theta)$ between the two regimes $\omega < \epsilon$ and $\omega > \epsilon$.

(b) When $T_{\rm eff} \neq 0$, the mechanical analogy deduced in (a) becomes a statistical physics problem in which θ is the coordinate of a Brownian particle moving on the 'landscape' $V(\theta)$. When the stable fixed point θ^* found in (a) exists and the $T_{\rm eff}$ is appropriately small, the particle exhibits noisy small-amplitude fluctuations around θ^*. Calculate the autocorrelation function $\langle \theta(t)\theta(t+\tau) \rangle$ in the limit of large t in that regime, expressing your answer in terms of ω, ϵ, and $T_{\rm eff}$. As the temperature is increased there will be an ever greater probability of crossing over potential energy barriers. Using general arguments about thermally-assisted hopping over potential barriers, deduce the relative probability $P = p_+/p_-$ of hopping from θ^* to $\theta^* + 2\pi$ versus from θ^* to $\theta^* - 2\pi$ when $\omega \ll \epsilon$.

END OF PAPER