UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2015 1:30 pm to 4:30 pm

PAPER 74

FLUID DYNAMICS OF THE ENVIRONMENT

You may attempt ***ALL*** *questions,*
although full marks can be achieved by good answers to ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

Completed answers are preferred to fragments. The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Derive the linearized equations governing small amplitude internal gravity waves in an otherwise quiescent stratified fluid of constant buoyancy frequency N_0. Determine the dispersion relation for plane waves with frequency ω and give an expression for the phase velocity $\mathbf{c}_p$. What is the geometric relationship between the phase velocity and the group velocity $\mathbf{c}_g$?

(b) Explain how the WKB approximation can be used to determine the behaviour of an internal wave in a stratification when $N = N(z)$, detailing any limitations and assumptions. Suppose a non-uniform stratification is characterised by the height-dependent buoyancy frequency

$$N(z) = \sqrt{(16z)^2 + 1}. \qquad (\star)$$

For a wave with $\omega = 1$, use ray tracing under the WKB approximation to determine the possible rays passing through $(x, z) = (0, 1)$. What happens as the ray approaches $z = 0$?

(c) Consider a trapezoidal basin described by $0 \leqslant x \leqslant 3(z + 2)$ for $0 \leqslant z \leqslant 1$ that contains a stratified fluid with buoyancy frequency $N(z)$ given by $(\star)$. Use ray tracing for an internal gravity wave of frequency $\omega = 1$ to determine where an upward propagating packet of energy released from $(x_0, 0)$ will next reflect from the lower boundary. [You may assue that the packet undergoes exactly three reflections before reaching the lower boundary again.] Hence or otherwise determine the internal wave attractor that exists for this system. In what direction must the energy packet propagate around the domain to form an attractor? What happens to energy packets initially propagating in the opposite direction? For a real fluid, what can limit the energy density achieved by the attractor?

2

Consider a layer of fluid of depth $h(x,t)$, velocity $u(x,t)$ and density $\rho(x,t)$ beneath a deep quiescent ambient fluid of density ρ_0. The layer is in a channel of width b that varies with height as $b(z) = \beta z$ for $z > 0$ and constant β. The channel is aligned with the x axis and z is upwards. The density of the layer is given by $\rho = (1-\phi)\rho_0 + \phi\rho_p$, where $\phi(x,t)$ is the volumetric concentration of particles of density ρ_p $(> \rho_0)$. The particles sediment relative to the fluid with a vertical velocity $w = w_0 f(\phi)$.

(a) Describe briefly the Boussinesq and shallow water assumptions and how they may apply to this flow. Under what conditions will the particle concentration remain approximately uniform across the depth of the layer? Explain how sedimentation can still occur.

(b) Derive the shallow water equations for the flow in this layer assuming $\phi \ll 1$ and the volume of the layer is conserved. Assume also that the particle concentration remains uniform across the width and depth of the layer, that sedimention is the same as for horizontal boundaries, and that no secondary circulations develop. Determine the characteristics of this flow and the ordinary differential equations (for u, ϕ and c) describing the changes along these characteristics. (You need not solve these ordinary differential equations.)

(c) Suppose that at $t = 0$ the layer is confined to the region $0 \leqslant x \leqslant L$ with layer depth h_0 and particle concentration ϕ_0. The dam at $x = L$ that contained the layer fails at $t = 0$. Explain why the shallow water equations are not appropriate for the front of the current that develops. Give a suitable condition for the speed of the front. Derive an integral model for the late time evolution of the current for the case $f(\phi) = 1 - \phi$ and determine the run-out length of the current.

3

Consider a bidisperse laminar suspension comprising particles with concentrations ϕ_1 and ϕ_2 that are identical apart from their radii, r_1 and r_2. There is no particle diffusion and the suspension, in a container of height H, is quiescent apart from particle settling and slow mixing. Particle settling is independent for each species and is governed by $W(\phi_i, r_i) = W_s(r_i)(1 - \phi_i/\phi_{max})$, where $W_s(r_i)$ is the settling velocity for isolated particles of radius r_i, and $\phi_i \leqslant \phi_{max}$. Consider the settling process at the bottom as an instantaneous transition to $\phi = \phi_{max}$ in the deposit.

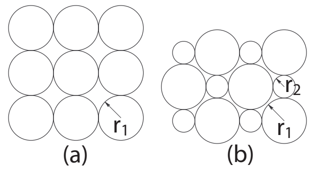

Figure 1: (a) Square packing. (b) Alternate packing.

(a) Determine the packing fraction for a square 2D packing of uniform cylinders with radius r_1 (see figure 1a) and the packing fraction for an alternate packing of equal number of cylinders of radii r_1 and r_2 (see figure 1b). What ratio r_1/r_2 gives the highest packing fraction?

(b) Change to spherical particles in a 3D geometry. Determine the ratio $W_s(r_1)/W_s(r_2)$ for both low and high particle Reynolds numbers Re_p.

(c) For low Re_p, consider a suspension containing only particles with radius $r_1 = a$ and initial concentration $\phi_1 = \phi_{max}/8$, and assume settling occurs in a perfect square arrangement (the 3D equivalent of figure 1a) where a fully settled particle packing satisfies $h_c/H = \phi_1/\phi_{max}$, with h_c the thickness of the deposit. Derive the general shock condition. Calculate the shock velocity U_1 between the cleared fluid and the settling suspension, and the shock velocity U_2 between the settling suspension and the settled particles at the bottom. Show that the process comes to rest after $t_c = H/W_s(a)$ when the particles reached a height $h_c = H/8$. Sketch a phase-space diagram of time t_c as a function of height h_c with the shocks drawn in.

(d) Change now to a bidisperse settling suspension of both large ($r_1 = a$) and small ($r_2 = a/2$) particles with uniform initial concentrations $\phi_1 = \phi_2 = \phi_{max}/8$. Assume the particle packing satisfies $h_2/H = (\phi_1 + \phi_2)/\phi_{max}$. Show that three shocks form initially with velocities $\hat{U}_1, \hat{U}_2, \hat{U}_3$. Show that $\hat{U}_3 = -\frac{35}{192} W_s(a)$ and determine $\hat{U}_1$ and $\hat{U}_2$. Show that at $t_1 = \frac{192}{203} \frac{H}{W_s(a)}$ two of the shocks merge, after which time only the small particles remain in suspension. For time $t > t_1$, determine the speed $\hat{U}_4$ of the merged shock. Sketch the evolution of the system in a phase-space diagram indicating the position of the shocks, t_1 and h_1, and also time t_2 and height h_2 of a fully settled suspension. Sketch the composition (as a function of height) of the final deposit in terms of percentage of large and small particles.

4

A time-dependent turbulent plume rises in the positive z direction from a line source located along $(x, y, z) = (0, y, 0)$ in a homogeneous quiescent environment of density ρ_0. The velocity and density fields within the plume vary as $\hat{w}(x, z, t) = W(z, t)f(x/b)$ and $\hat{\rho}(x, z, t) = \rho_0 + (\rho(z, t) - \rho_0)f(x/b)$, where W and ρ are the centreline velocity and density, respectively. Here, the profiles are assumed to be triangular such that

$$f\left(\frac{x}{b}\right) = \begin{cases} 1 - \left|\frac{x}{b}\right| & \left|\frac{x}{b}\right| \leqslant 1, \\ 0 & \text{otherwise,} \end{cases}$$

and $b(z, t)$ is the half-width of the plume.

(a) Determine, per unit length in the y direction, the plume volume, mass, momentum and buoyancy in terms of W, ρ and b. Determine also the fluxes (again per unit length) of volume V, mass Q, momentum M and buoyancy F.

(b) Explain the idea behind 'Batchelor entrainment' and the entrainment coefficient α, and give a suitable expression for entrainment into the plume.

(c) Derive the time-dependent equations of motion for the plume in terms of W, ρ and b. Hence or otherwise show that a Boussinesq plume satisfies

$$\frac{4}{3}\frac{\partial}{\partial t}\left(\frac{Q^2}{M}\right) + \frac{\partial Q}{\partial z} = 3\alpha\frac{\rho_0 M}{Q},$$

$$\frac{\partial Q}{\partial t} + \frac{\partial M}{\partial z} = \frac{FQ}{M},$$

$$\frac{\partial}{\partial t}\left(\frac{FQ}{M}\right) + \frac{\partial F}{\partial z} = 0.$$

(d) Determine the time-dependent similarity solution for Q, M and F for $t \geqslant 0$.

END OF PAPER