UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2015 9:00 am to 12:00 pm

PAPER 73

SLOW VISCOUS FLOW

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State the Papkovich–Neuber representation for the velocity and pressure in Stokes flow. Use this representation, explaining your choice of trial harmonic potentials, to determine the velocity field due to a point force of magnitude $\mathbf{F}$ in unbounded fluid.

(b) A constant force $\mathbf{F} = 6\pi\mu a\mathbf{V}$ acts on a rigid sphere of radius a at position $\mathbf{x} = \mathbf{X}(t)$. A second rigid sphere of radius a is held stationary at $\mathbf{x} = \mathbf{0}$.

Assuming that $a \ll R$, where $R = |\mathbf{X}|$, find the *leading-order* approximations to the force and the couple that must be applied to the second sphere to keep it stationary. State the order of the next correction to the force, and state where it comes from.

Deduce the leading-order correction to the velocity of the first sphere due to the presence of the second sphere, and state the order of the next correction. Show also that the first sphere rotates with angular velocity

$$\boldsymbol{\Omega} = -\frac{9a^2}{16}\frac{\mathbf{V} \wedge \mathbf{X}}{R^4} + O(Va^4/R^5).$$

[*You may assume the Faxén formulae*

$$\mathbf{U} = \frac{\mathbf{F}}{6\pi\mu a} + \mathbf{u}_\infty + \frac{a^2}{6}\nabla^2\mathbf{u}_\infty\ , \quad \boldsymbol{\Omega} = \frac{\mathbf{G}}{8\pi\mu a^3} + \frac{1}{2}\boldsymbol{\omega}_\infty\ ,$$

but should explain how you apply them.]

(c) Cartesian coordinates are defined for the problem in part (b) such that $\mathbf{X}(t) = (X(t), Y(t), 0)$ and $\mathbf{V} = (V, 0, 0)$, where $V > 0$. Explaining any approximations, show that the path of the first sphere satisfies

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = -\frac{27a^2}{16}\frac{XY}{R^4}\ .$$

As $t \to -\infty$, it is given that $Y(t) \to Y_\infty$, where $Y_\infty \gg a$. Find leading-order approximations to (i) the maximum value of the small deflection $Y(t) - Y_\infty$ and (ii) the angle the sphere has rotated by as $t \to \infty$.

What happens to $Y(t)$ as $t \to \infty$ and why? Would your answer be the same if $Y_\infty = \frac{1}{100}a$? Justify your answer briefly.

2

Write down the extensional-flow equations governing the evolution of a long thin axisymmetric column of fluid with viscosity μ, radius $a(z,t)$ and axial velocity $w(z,t)$, under an axial body force f and external pressure $p_{ext}(z)$.

Such a column of fluid, with density $\rho + \Delta\rho$, is placed on the axis of a vertical cylindrical container of slightly larger radius R. The column sits on the closed base of the container, $z = 0$. The thin annular gap, of width $h(z,t) \ll R$, between the fluid column and the container is filled with a second fluid of much smaller viscosity $\lambda\mu$ and density ρ. The fluid column slumps axisymmetrically due to its greater density.

Use physical reasoning and scaling arguments to show that the variation in w across the fluid column is negligible provided $\lambda \ll h^2/a^2$. Assume below that this condition holds.

Use lubrication theory and mass conservation to determine the (modified) pressure gradient in the annular gap in terms of a and w. Deduce dimensional evolution equations for the column.

For the case $\lambda = 0$, and an initially cylindrical column of radius a_0 and height L_0, show that the vertical velocity at the top of the column at $t = 0$ is $-\Delta\rho g L_0^2/6\mu$.

For the case $\lambda \neq 0$, find scales $\hat{z}$, $\hat{t}$ and $\hat{w}$ such that $\phi = a^2/R^2$ and $W = w/\hat{w}$ satisfy dimensionless equations of the form

$$\frac{1}{\phi}\frac{\partial}{\partial Z}\left(\phi\frac{\partial W}{\partial Z}\right) = 1 + \frac{\alpha(\phi)W}{\alpha_0} \tag{1}$$

$$\frac{\partial \phi}{\partial T} + \frac{\partial}{\partial Z}(\phi W) = 0, \tag{2}$$

where the dimensionless function $\alpha(\phi)$ should be defined and $\alpha_0 = \alpha(a_0^2/R^2)$.

For an initially cylindrical column of radius a_0 and height $L_0 = \Lambda_0\hat{z}$, determine the dimensionless vertical velocity $W(Z,0)$ at $T = 0$ as a function of the dimensionless height Λ_0. Deduce the rate of thickening of the column $\partial\phi/\partial T$ at $T = 0$.

Find and sketch the limiting forms of these results as functions of Z for each of the cases $\Lambda_0 \gg 1$ and $\Lambda_0 \ll 1$. Describe the dominant physical balances in the two cases, and give a physical interpretation of the meaning of the vertical scale $\hat{z}$.

3

Insoluble surfactant with concentration $C(x,t)$ resides on the surface of a thin layer of fluid of thickness $h(x,t)$, viscosity μ and density ρ that lies on a rigid horizontal surface. Diffusion of surfactant is negligible, and the coefficient of surface tension is given by $\gamma(C) = \gamma_0 - AC$, where γ_0 and A are constants. The variations of h and C are such that lubrication theory is applicable throughout.

Explain why

$$\frac{\partial C}{\partial t} + \frac{\partial [u(h)C]}{\partial x} = 0,$$

where $u(h)$ is the horizontal velocity at the surface. Show that

$$\frac{\partial h}{\partial t} = \frac{A}{2\mu}\frac{\partial}{\partial x}\left(h^2\frac{\partial C}{\partial x}\right) + \frac{\rho g}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right) - \frac{1}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial}{\partial x}\left(\gamma\frac{\partial^2 h}{\partial x^2}\right)\right), \qquad (*)$$

and obtain the corresponding equation for the evolution of C.

Assume that the hydrostatic and capillary pressure gradients are both negligible. A fixed mass $M = \int C\,\mathrm{d}x$ of surfactant is released at $x = 0$ and $t = 0$ onto a layer that initially has uniform thickness h_0 and $C = 0$. Use scaling arguments to show that the extent $-x_N \leqslant x \leqslant x_N$ of the spreading pool of surfactant satisfies $x_N(t) \propto t^{1/3}$ and also determine the dependence on the other dimensional parameters.

Deduce the form of the similarity solution and derive two ordinary differential equations and two integral constraints that are satisfied by the dimensionless similarity functions $H(\eta)$ and $\Gamma(\eta)$ over the range $0 \leqslant \eta \leqslant \eta_N$ (assuming symmetry about $\eta = 0$). Solve the differential equations to show that H and Γ are linear functions of η, and use the integral constraints to show that

$$x_N = (6MAh_0 t/\mu)^{1/3}, \qquad \text{and } h = 2h_0 \text{ at } x = x_{N-}.$$

Explain why the hydrostatic and capillary terms in $(*)$ cannot both be negligible near $x = x_N$. Let Δ be the width of the region where at least one of these terms is significant. Use scaling arguments to show that when $g = 0$

$$\Delta \sim (\gamma_0 h_0^2 x_N^2/AM)^{1/3} \propto t^{2/9}\ .$$

[*Assume that* $\partial C/\partial x$ *continues to scale like* M/x_N^2.]

Find the corresponding result for Δ when $g \neq 0$ and $\Delta^2 \gg \gamma_0/g$. Deduce that there is a time t^* when $\Delta \sim x_N$. Explain briefly what happens to the fluid layer when $t \gg t^*$.

4

A long cylindrical tube of length L and radius $a \ll L$ is immersed in a large volume of viscous fluid. The tube is held fixed with its axis vertical, and is open at both ends so that it is both filled with and surrounded by fluid. A heavy sphere of radius $a(1 - \frac{1}{2}\epsilon)$, where $\epsilon \ll 1$, falls down the axis of the tube at velocity U.

Explaining any approximations made, show that the flux Q out of the bottom of the tube is related to the pressure difference ΔP across the sphere by

$$Q = \frac{\pi a^4 \Delta P}{8\mu L} \ . \tag{1}$$

Use lubrication theory to show that

$$\frac{\Delta P}{6\mu} = 2qI_3 - UI_2 \ , \tag{2}$$

$$\text{where} \ \ q = (\pi a^2 U - Q)/2\pi a \tag{3}$$

and the constants I_n are to be determined as multiples of $\epsilon^{\frac{1}{2}-n} \displaystyle\int_{-\infty}^{\infty} (1 + \xi^2)^{-n}\, \mathrm{d}\xi$.

By considering the forces acting on a suitable fluid control volume, show further that the upward force F on the sphere is given by

$$F = \pi a^2 \Delta P + 2\pi a\mu(4UI_1 - 6qI_2). \tag{4}$$

Let dimensionless variables be defined by

$$Q^* = \frac{Q}{\pi a^2 U}, \quad \Delta P^* = \frac{\Delta P a}{6\mu U}, \quad q^* = \frac{2q}{aU}, \quad F^* = \frac{F}{6\pi\mu a U}, \quad I_n^* = a^{n-1} I_n, \quad L^* = \frac{4L}{3a}.$$

Express (1)–(3) in terms of these variables and solve for Q^*, ΔP^* and q^*. Hence obtain

$$F^* = \frac{L^* I_3^* + (\frac{4}{3} I_1^* I_3^* - I_2^{*2})}{L^* + I_3^*}$$

explaining any approximations made. Deduce that F^* takes distinct asymptotic forms when (i) $L^* \ll \epsilon^{-1/2}$, (ii) $\epsilon^{-1/2} \ll L^* \ll \epsilon^{-5/2}$ and (iii) $L^* \gg \epsilon^{-5/2}$, and find the corresponding leading-order approximations.

By considering the size of Q^*, q^* and $F^*/\Delta P^*$, describe the dominant flow pattern and the dominant force balance in each of the three regimes.

Hence explain physically, without detailed calculation, why the force on the sphere decreases by a factor of 2 in one of the three regimes, but is unchanged in the other two, when there is a second sphere of the same radius also moving with velocity U down the tube.

END OF PAPER