UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 29 May, 2015 1:30 pm to 4:30 pm

PAPER 72

PERTURBATION AND STABILITY METHODS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Find the first **two** nonzero terms in the expansion for all real roots of the quintic equation

$$t^5 - \epsilon t^3 + \epsilon^3 = 0$$

as $\epsilon \to 0$.

(b) Consider the integral

$$f(x) = \int_0^1 (\ln t) \exp(\mathrm{i}xt) \mathrm{d}t$$

as $x \to \infty$. By suitably deforming the integration contour, show that

$$f(x) \sim -\frac{\mathrm{i}\ln x}{x} - \frac{\mathrm{i}\gamma + \pi/2}{x} + \frac{\exp(\mathrm{i}x)}{x^2} + O(x^{-3}) \ ,$$

where γ is Euler's constant. You are given that

$$\int_0^\infty \ln u \exp(-u) \mathrm{d}u = -\gamma \ .$$

(c) Consider

$$I(\lambda) = \int_{-\infty}^b \exp(\lambda\phi(t)) \mathrm{d}t \ ,$$

where

$$\phi(t) = t^3 - 2t^2 + t$$

and b is a real positive constant. Determine the first nonzero term in the expansion of I as $\lambda \to \infty$, being careful to identify the behaviours found for different values of b.

2

(a) Consider the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \epsilon f(x)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0 ,$$

subject to $x = \mathrm{d}x/\mathrm{d}t = 1$ when $t = 0$, where $\epsilon \ll 1$ and $f(x)$ is a given function. Find the leading-order approximation to $x(t;\epsilon)$ which is uniformly valid for $t \leqslant O(1/\epsilon)$ when (i) $f(x) = x^2 - 1$; (ii) $f(x) = \sin x$.

[Hint: in (ii) write $\sin x$ as a power series.]

(b) Consider the double integral

$$h(\lambda) = \int_A \alpha(x,y)\exp(\mathrm{i}\lambda\beta(x,y))\mathrm{d}x\mathrm{d}y$$

over the area A, and suppose that $\nabla\beta = 0$ at a single point in A. By using the method of stationary phase, determine the asymptotic behaviour of $h(\lambda)$ as $\lambda \to \infty$. [Hint: consider the matrix of second partial derivatives of β.]

3

The function $y(x)$ satisfies the differential equation

$$\varepsilon^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\varepsilon^2}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - 4x^2 y = 1\,,$$

where $0 < \varepsilon \ll 1$. Verfiy that a solution such that

$$y \to 0 \quad \text{as } x \to \infty \text{ with } \arg(x) \in [0, \tfrac{\pi}{2}),$$

is given, for appropriate choices of θ_1 and θ_2 (which should be specified), by

$$y = \frac{1}{4\varepsilon}\left(e^{x^2/\varepsilon} \int_{\infty\, e^{i\theta_1}}^{x} \frac{e^{-t^2/\varepsilon}}{t}\,\mathrm{d}t - e^{-x^2/\varepsilon} \int_{\infty\, e^{i\theta_2}}^{x} \frac{e^{t^2/\varepsilon}}{t}\,\mathrm{d}t \right),$$

where the limit $\infty\, e^{i\varphi}$ indicates that the integration contour tends to infinty along a line with argument φ. From this solution, find the leading term of the asymptotic expansion as $x \to \infty$ if $\arg(x) \in [0, \frac{\pi}{2})$. Identify the *anti-Stokes* lines, and the range of $\arg(x)$ for which the solution decays as $x \to \infty$.

Show that, formally, a consistent asymptotic solution for $0 < \varepsilon \ll 1$, assuming that it exists, is given by

$$y(x) \sim -\frac{1}{4x^2} \sum_{r=0}^{\infty} a_r \frac{\varepsilon^{2r}}{x^{4r}}$$

where the coefficients a_r are to be determined.

Optimally truncate this asymptotic expansion, and find an expression for the remainder when $\arg(x) = O(\varepsilon^{\frac{1}{2}})$. Comment on your answer.

Hints.

- You may find it helpful to consider the integral

$$J(\lambda, n) = \int_0^{\infty} \frac{t^{2n}}{1 - t^2} \exp(\lambda(1 - t))\,\mathrm{d}t\,.$$

- You may quote the result that

$$\int_0^{\infty} \frac{t^{2n}}{1 + t} \exp(\lambda(1 - t))\,\mathrm{d}t \sim \frac{1}{2}\left(\frac{\pi}{2n}\right)^{\frac{1}{2}} \exp(-\mu^2/8)\,,$$

  when

$$\lambda = 2n + i\mu n^{\frac{1}{2}} + \nu\,, \quad \mu = O(1),\ \nu = O(1) \text{ and } n \gg 1\,.$$

- Recall also that

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} \exp(-t^2)\,\mathrm{d}t, \quad \text{where } \mathrm{erf}(\infty) = 1\,.$$

4

Suppose that there is a steady two-dimensional incompressible unidirectional flow

$$\bar{\mathbf{u}} = (U(y), 0)\,,$$

in the half-space $y \geqslant 0$ above a rigid boundary. Assume dimensionless variables and that

$$\begin{aligned} U(y) &\sim y \quad \text{for} \quad 0 \leqslant y \ll 1 \quad (\text{and hence that } U(0)=0)\,; \\ U(y) &\to 1 \quad \text{as} \quad y \to \infty\,. \end{aligned}$$

Consider the linear stability of this flow by writing

$$(\bar{u}, \bar{v}, \bar{p}) = (U(y), 0, 0) + \varepsilon(u(y), v(y), p(y))\, \mathrm{e}^{i\alpha(x-ct)} + \dots\,,$$

where $0 < \varepsilon \ll 1$, α is the (real) streamwise wavenumber, c is the (complex) wavespeed, and u, v and p represent the velocity and pressure perturbations. On substituting into the conservation of mass and Navier-Stokes equations, and linearising, the perturbation equations are found to be

$$\begin{aligned} i\alpha u + v' &= 0\,, & (1)\\ i\alpha(U-c)u + U'v &= -i\alpha p + \tfrac{1}{R}(u'' - \alpha^2 u)\,, & (2)\\ i\alpha(U-c)v &= -p' + \tfrac{1}{R}(v'' - \alpha^2 v)\,, & (3) \end{aligned}$$

where R is the Reynolds number. Appropriate boundary conditions are

$$\begin{aligned} u = v = 0 \quad &\text{on} \quad y = 0\,, \\ u \to 0\,,\ v \to 0 \quad &\text{as} \quad y \to \infty\,. \end{aligned}$$

Derive an equation satisfied by the Laplacian of the pressure perturbation, and state an expression for the value of p on $y = 0$.

Assume that the Reynolds number, R, is large and that the streamwise wavenumber, α, is small (but not too small); specifically, assume that $R^{-\frac{1}{7}} \ll \alpha \ll 1$. Make the hypothesis that a thin "lower" layer of thickness δ exists adjacent to the boundary $y = 0$, and that within this lower layer (a) both terms in equation (1) balance, and (b) all terms, *except the last term*, in equation (2) balance. Explain why a suitable inner scaling is $y = \delta Y = (\alpha R)^{-\frac{1}{3}} Y$, and why a suitable scaling for the wavespeed is $c = \delta C$. Also deduce scalings for v and p in terms of α and R on the assumption that $u = O(1)$ in the lower layer. Hence show that

$$i(Y - C)u_Y = u_{YYY}\,.$$

Solve for u and p within the lower layer, and find their values as $Y \to \infty$.

Match the lower-layer solution with a leading-order solution for $y = O(1)$. How does the pressure vary across this $y = O(1)$ "middle" layer? Find an expression for v as $y \to \infty$. Deduce that an "upper" layer, say with scaling $y = \Delta z$ (where Δ is to be identified), is required. Solve in this upper layer and match to the middle layer. By assuming that $\alpha = kR^{-\frac{1}{4}}$, derive a dispersion relation between k and C.

Hint. Recall that the Airy function, $\mathrm{Ai}(\zeta)$, satisfies

$$\mathrm{Ai}''(\zeta) - \zeta \mathrm{Ai}(\zeta) = 0\,,$$

and that $\mathrm{Ai}(\zeta)$ decays exponentially to zero as $|\zeta| \to \infty$ if $|\arg(\zeta)| < \pi/3$.

END OF PAPER