UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2015 9:00 am to 11:00 am

PAPER 71

DISTRIBUTION THEORY AND APPLICATIONS

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $X \subset \mathbf{R}^n$ be open. Define the spaces $\mathcal{E}(X)$ and $\mathcal{E}'(X)$, specifying the notion of convergence in each.

Show that a linear map $u : \mathcal{E}(X) \to \mathbf{C}$ belongs to $\mathcal{E}'(X)$ if and only if $\langle u, \varphi_m \rangle \to 0$ for every sequence $\{\varphi_m\}_{m \geqslant 1}$ that tends to zero in $\mathcal{E}(X)$ (*you may assume that there exist compact sets* $\{K_m\}_{m \geqslant 1}$ *with* $K_m \subset K_{m+1}$ *such that* $X = \bigcup_{m \geqslant 1} K_m$).

For each $u \in \mathcal{E}'(X)$, show there exists a *finite* collection of continuous functions $\{f_\alpha\}$, each with compact support contained in X, such that

$$u = \sum_\alpha \partial^\alpha f_\alpha \quad \text{in } \mathcal{E}'(X).$$

Can a general element of $\mathcal{D}'(X)$ be written as a finite sum of distributional derivatives of continuous functions? Give a proof or counterexample.

2

Define the Schwartz space of functions $\mathcal{S}(\mathbf{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbf{R}^n)$. Show that the Fourier transform defines a continuous isomorphism $\mathcal{F} : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$. Hence show that the Fourier transform extends to a continuous isomorphism on the space of tempered distributions.

Define the convolution between $\mathcal{S}(\mathbf{R})$ and $\mathcal{S}'(\mathbf{R})$. Show that if $u \in \mathcal{S}'(\mathbf{R}^n)$ and $\varphi \in \mathcal{S}(\mathbf{R}^n)$ then $u * \varphi \in \mathcal{S}'(\mathbf{R}^n)$ and $(u * \varphi)\hat{} = \hat{u}\hat{\varphi}$.

For $\varphi \in \mathcal{S}(\mathbf{R})$ define the operator $\mathcal{H}$ by

$$\mathcal{H}\varphi(x) := \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{\varphi(y)}{x-y} \, \mathrm{d}y.$$

Establish the following properties of $\mathcal{H}$ as an operator on $\mathcal{S}(\mathbf{R})$:

(a) $\|\mathcal{H}\varphi\|_{L^2} = \|\varphi\|_{L^2}$, where $\|f\|_{L^2}^2 := \int |f(x)|^2 \, \mathrm{d}x$.

(b) $\mathcal{H}\varphi$ is smooth and $(\mathcal{H}\varphi)'(x) = (\mathcal{H}\varphi')(x)$.

(c) The large $|x|$ behaviour of $\mathcal{H}\varphi$ is given by

$$\mathcal{H}\varphi(x) = \frac{\hat{\varphi}(0)}{\pi x} + o\left(\frac{1}{x}\right), \quad |x| \to \infty$$

[Hint: you may find it helpful to recall $\hat{H} = \pi\delta_0 - \mathrm{i}\,\mathrm{p.v.}(1/x)$ *where* H *denotes the Heaviside function.]*

3

What does it mean for a function $\Phi = \Phi(x, \theta)$ to be a *phase function*? Define the space of symbols $\mathrm{Sym}(X; \mathbf{R}^k; N)$, where $X \subset \mathbf{R}^n$ is open. Prove the following results

(i) If $a \in \mathrm{Sym}(X, \mathbf{R}^k, N)$ then $D_x^\alpha D_\theta^\beta a \in \mathrm{Sym}(X, \mathbf{R}^k; N - |\beta|)$.

(ii) If $a_i \in \mathrm{Sym}(X, \mathbf{R}^k; N_i)$ for $i = 1, 2$ then $a_1 a_2 \in \mathrm{Sym}(X, \mathbf{R}^k; N_1 + N_2)$.

Now suppose Φ is a phase function and $a \in \mathrm{Sym}(X, \mathbf{R}^k; N)$. Describe how the oscillatory integral

$$I_\Phi(a) = \int e^{\mathrm{i}\Phi(x,\theta)} a(x, \theta)\, \mathrm{d}\theta$$

can be used to define a linear map from $\mathcal{D}(X)$ to $\mathbf{C}$. You may assume the resulting linear map belongs to $\mathcal{D}'(X)$.

Define the singular support of an element of $\mathcal{D}'(X)$. Prove that the singular support of $I_\Phi(a)$ is contained in the set

$$\{x : \nabla_\theta \Phi(x, \theta) = 0, \text{ for some } \theta \in (\mathbf{R}^k \setminus \{0\}) \cap \mathrm{supp}\, a(x, \theta)\}$$

For $(x, \theta) \in \mathbf{R}^n \times \mathbf{R}^n$, show that the oscillatory integral

$$\frac{1}{(2\pi)^n} \int \theta^\alpha e^{\mathrm{i}x \cdot \theta}\, \mathrm{d}\theta$$

coincides with a well known element of $\mathcal{D}'(\mathbf{R}^n)$.

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