UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2015 9:00 am to 11:00 am

PAPER 70

BOUNDARY VALUE PROBLEMS FOR LINEAR PDES

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let the complex valued function $q(x,t)$ satisfy

$$iq_t + q_{xx} = 0, \quad 0 < x < \infty, 0 < t < T, \tag{1}$$

where T is a positive finite constant.

(a) Express q as an integral in the complex k-plane.

(b) Consider (1) together with

$$q(x,0) = q_0(x), \quad 0 < x < \infty, \tag{2a}$$
$$q(0,t) = g_0(t), \quad 0 < t < T, \tag{2b}$$

where q_0 and g_0 have sufficient smoothness, q_0 has sufficient decay as $x \to \infty$, and $q_0(0) = g_0(0)$. Obtain a uniformly convergent integral representation for $q(x,t)$ in terms of the given data, and verify that $q(x,t)$ satisfies (1) and (2).

(c) Use (a) and $q = u + iv, u, v$ real, to solve the following initial-boundary value problem for the elastic wave eq:

$$\begin{array}{lll} u_{tt} + u_{xxxx} = 0, & 0 < x < \infty, & 0 < t < T \\ u(x,0) = u_0(x), & u_t(x,0) = u_1(x), & 0 < x < \infty \\ u(0,t) = \tilde{u}_0(t), & u_{xx}(0,t) = \tilde{u}_1(t), & 0 < t < T, \end{array}$$

where the given functions $u_0, u_1, \tilde{u}_0, \tilde{u}_1$ have sufficient smoothness, u_0, u_1 have sufficient decay as $x \to \infty$, and $u_0(0) = \tilde{u}_0(0), u_0''(0) = \tilde{u}_1(0), \tilde{u}_0' = u_1(0)$.

2

Let the real valued function $q(x,y)$ satisfy the following boundary value problem,

$$\begin{aligned}
&q_{xx}+q_{yy}=0, && 0<x<\infty, \quad 0<y<\ell,\\
&-q_y(x,0)+\gamma q(x,0)=0, && 0<x<\infty\\
&q_y(x,\ell)+\gamma q(x,\ell)=0, && 0<x<\infty\\
&q_x(0,y)=g(y), && 0<y<\ell
\end{aligned}$$

where $\ell>0,\gamma$, are constants, and $g(y)$ is a given real, smooth function satisfying $g(y)=g(\ell-y)$.

(a) Let subscripts 1,2,3 refer to the sides

$$\{y=0, \quad 0<x<\infty\}, \quad \{x=0, \quad 0<y<\ell\}, \quad \{y=\ell, 0<x<\infty\}, \tag{1}$$

respectively. Let $\{G_j(k)\}_{j=1}^3$ denote the appropriate transforms of q_z, i.e.

$$G_j(k)=\int_{z_j}^{z_{j+1}} e^{-ikz} q_z dz, \quad z_1=\infty, \quad z_2=0, \quad z_3=i\ell, \tag{2}$$

where $z_4 \equiv z_1$ and set

$$\psi(-ik)=\frac{1}{2}\int_0^\infty e^{-ikx} q(x,0)dz, \quad \phi(k)=\frac{1}{2}\int_0^\ell e^{ky} q_y(0,y)dy\,.$$

Express $G_1(k),G_3(k)$, in terms of $q(0,0)$ and the unknown function $\psi(-ik)$. Express $G_2(k)$ in terms of the unknown function $\phi(k)$.

(b) Use the global relation to express both $\psi(-ik)$ and $\phi(k)$ in terms of $\psi(ik)$.

(c) Use the integral representation for $q(x,y)$

$$q_z=\frac{1}{2\pi}\left(\int_0^\infty e^{ikz}G_1(k)dk+\int_0^{i\infty} e^{ikz}G_2(k)dk+\int_0^{-\infty} e^{ikz}G_3(k)dk\right) \tag{3}$$

to show that $\psi(ik)$ does not contribute to the solution.

3

Obtain two global relations associated with real solutions of the modified Helmholtz equation.

Use these global relations to derive a semi-analytical scheme to integrate numerically the Dirichlet problem of the above equation formulated in the interior of a square with corners at $(-1,1),(-1,-1),(1,-1),(1,1)$. In particular, verify that there exists a choice of collocation points associated with each side, such that the effect on the other sides becomes small as the value of the collocation parameter becomes large.

END OF PAPER