UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 3 June, 2015 1:30 pm to 4:30 pm

PAPER 7

SEMIGROUPS OF OPERATORS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Suppose that μ is a Borel measure on a locally compact metric space (X, d) and that f is a non-negative continuous function on X. If $f \in C_0(X)$ (the space of continuous functions f on X for which $f(x) \to 0$ as $x \to \infty$) or if $f \in L^2_\mu$, and if $t \geqslant 0$ let $P_t(f) = e^{-k(x)t}f(x)$, where k is a non-negative continuous function. In each case, show that $(P_t)_{t \geqslant 0}$ is a contraction semigroup.

In each case, determine the infinitesimal generator L of the semigroup, determine its domain $D(L)$ and show directly that L is a closed operator.

Explain how such semi-groups can be used to investigate the heat semigroup acting on L^2_λ and the Ornstein-Uhlenbeck semigroup acting on L^2_γ (you need only consider the one-dimensional cases). In each case, determine the spectrum of L and show that their infinitesimal generators are self-adjoint.

Does the heat semigroup satisfy a Poincaré inequality? Does the Ornstein-Uhlenbeck semigroup satisfy a Poincaré inequality? Justify your answers.

2

Suppose that (X, d) is a compact metric space. What is a *Feller semigroup* of operators on $C(X)$?

Suppose that $(P_t)_{t \geqslant 0}$ is a contraction semigroup on $C(X)$ with infinitesimal generator L. Show that if $1 \in D(L)$ and $L(1) = 0$ then $(P_t)_{t \geqslant 0}$ is a Feller semigroup.

Suppose now that $(P_t)_{t \geqslant 0}$ is a Feller semigroup on $C(X)$ with infinitesimal generator L. Establish the existence of transition probabilities.

What is an *invariant* probability measure on X? Show that if μ is an invariant probability measure on X, if $1 \leqslant p < \infty$ and if $f \in C(X)$ then $\|P_t(f)\|_p \leqslant \|f\|_p$ for $t > 0$, and that $\|P_t(f) - f\|_p \to 0$ as $t \to 0$.

Suppose that f and f^2 are in $D(L)$. Show that $L(f^2) \geqslant 2fL(f)$.

Let ν be a Borel probability measure on X. If $f \in C(X)$, let

$$\phi_n(f) = \frac{1}{n}\int_0^n \left(\int_X P_s(f)\, d\nu\right) ds.$$

Explain briefly why there exist a subsequence (ϕ_{n_k}) and a Borel probability measure μ such that $\phi_{n_k}(f) \to \int_X f\, d\mu$ as $k \to \infty$, for all $f \in C(X)$. Show that μ is an invariant probability measure.

3

Suppose that A is a symmetric closed linear operator on a Hilbert space H, with dense domain. What does it mean to say that A is *self-adjoint*? Show that if A is self-adjoint then $\sigma(A) \subseteq \mathbf{R}$.

Suppose that T is a bounded self-adjoint operator on a Hilbert space H. Define the *numerical range* $W(T)$ and the *numerical radius* $w(T)$ of T. Show that $\sigma(T) \subseteq \overline{W(T)} \subseteq \mathbf{R}$.

Show that $\alpha = \inf W(T)$ and $\beta = \sup W(T)$ are approximate eigenvalues of T.

Suppose that A is a self-adjoint closed linear operator on H with dense domain. Show that A is positive semi-definite if and only if $\sigma(A) \subseteq [0, \infty)$.

4

Let G be the group D_2^d, with Haar measure μ. Define the *Bernoulli functions* ϵ_i and *Walsh functions* w_A on G, and show that the Walsh functions form an orthonormal basis for $L^2(G)$.

If $\omega, \eta \in G$, set $\omega \sim \eta$ if $\omega_i \neq \eta_i$ for exactly one index i. If $f \in L^2(G)$, set

$$L(f)(\omega) = \frac{1}{2} \sum_{\eta \sim \omega} (f(\eta) - f(\omega)).$$

Detremine the eigenvectors and eigenvalues of L, and show that L is a negative semi-definite operator on $L^2(G)$.

By applying L to a suitable function f show that if $a_1, \ldots, a_d$ are elements of a normed space $(E, \|.\|)$ then

$$\int_G \| \sum_{i=1}^d a_i \epsilon_i \|^2 \, d\mu \leqslant 2 \left(\int_G \| \sum_{i=1}^d a_i \epsilon_i \| \, d\mu \right)^2 .$$

Suppose that $\eta_1, \ldots, \eta_d$ are independent random variables, each uniformly distributed on $\mathbf{T} = \{z : |z| = 1\}$, and that $a_1, \ldots, a_d$ are complex numbers. Show that

$$\sum_{i=1}^d |a_i|^2 \leqslant 2 \left(\mathbf{E} | \sum_{i=1}^d a_i \eta_i | \right)^2 .$$

5

Let γ denote standard Gaussian measure on $\mathbf{R}$. Define the creation operator a^+, the annihilation operator a^- and the number operator N. Define the Hermite polynomials, and show that they are the eigenfunctions of N. (You may assume that the polynomial functions are dense in $L^2(\gamma)$.)

Show that N is positive semi-definite. How is N used to define the Ornstein-Uhlenbeck semigroup $(P_t)_{t\geqslant 0}$? Show that $P_t(f) \to \int f\,d\gamma$ as $t \to \infty$, and that $dP_t(f)/dx = e^{-t}P_t(df/dx)$.

Suppose that f is a differentiable function with continuous bounded derivative, that $f(x) > \epsilon$ for all $x \in \mathbf{R}$ and that $\|f\|_1 = 1$. Find an expression for the energy $\mathcal{E}_\gamma(f)$. Show that $\mathrm{Ent}_\gamma(f^2) \leqslant 2\mathcal{E}_\gamma(f)$.

(You may assume that if g is a bounded continuous function, then $(P_t(g))^2 \leqslant P_t(g^2/f).P_t(f)$.)

END OF PAPER