UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2015 1:30 pm to 4:30 pm

## PAPER 69

## APPROXIMATION THEORY

*Attempt no more than* ***FOUR*** *questions.*

*There are* ***SIX*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

i) State the Korovkin theorem in its most general form, i.e., give a sufficient condition for the uniform convergence $U_n(f) \to f$ for a sequence $(U_n)_{n=1}^{\infty}$ of positive linear operators $U_n : C(K) \to C(K)$, where $K$ is compact. As a corollary derive a sufficient condition for the periodic case when $C(K) = C(\mathbb{T})$, the space of $2\pi$-periodic continuous functions.

ii) For a $2\pi$-periodic function $f \in C(\mathbb{T})$, let $s_n(f)$ be its partial Fourier sum of degree $n$, and let $\sigma_n(f) = \frac{1}{n}\sum_{i=0}^{n-1} s_i(f)$ be its Fejer sum of degree $n-1$.

Using the Korovkin theorem prove that $\sigma_n(f) \to f$ as $n \to \infty$ for all $f \in C(\mathbb{T})$. You may use the integral representation for the Fourier sum

$$s_n(f,x) = \frac{1}{\pi}\int_{\mathbb{T}} D_n(x-t) f(t)\, dt, \qquad D_n(x) = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x},$$

and the fact that $s_n$ is an orthogonal projector onto $\mathcal{T}_n$.

iii) Prove that the only positive linear operator $U : C(\mathbb{T}) \to C(\mathbb{T})$ such that

$$U(p_i) = p_i \qquad \text{where} \quad p_i \in \{1, \sin x, \cos x\}$$

is the identity operator.

**2**

A) Define a multiresolution analysis of $L_2(\mathbb{R})$ with a generator $\phi$ and explain how it is related to existence of an orthonornal wavelet $\psi$.

B) Prove that the following properties of $\phi$

$$1) \quad \phi(x) = \sum_n a_n \phi(2x - n), \qquad 2) \quad \{\phi(\cdot - n)\}_{n\in\mathbb{Z}} \quad \text{is an orthonormal sequence}$$

are equivalent to

$$1') \qquad f(2t) = m(t) f(t)\,, \qquad m(t) = \tfrac{1}{2}\sum_n a_n e^{-int},$$

$$2') \qquad \sum |f(t + 2\pi k)|^2 \equiv 1 \text{ a.e.}$$

where $f$ is the Fourier transform of $\phi$, i.e., $f(t) = \widehat{\phi}(t) = \int_{\mathbb{R}} \phi(x) e^{-ixt}\, dx$.

C) Verify that conditions $1') - 2')$ are fulfilled for the function $f = \widehat{\phi}$ defined as

$$f(t) = \begin{cases} 1, & t \in [-\pi, \pi) \\ 0, & \text{otherwise} \end{cases}$$

Using the inverse Fourier transform or otherwise, determine the corresponding generator $\phi$.

**3**

Let $j_n$ be the Jackson operator, i.e., for a $2\pi$-periodic function $f$ from $C(\mathbb{T})$,

$$j_n(f, x) := \int_{-\pi}^{\pi} f(x - t) J_n(t)\, dt, \quad J_n(t) := \frac{3}{2\pi n(2n^2 + 1)} \frac{\sin^4 \frac{nt}{2}}{\sin^4 \frac{t}{2}}\,, \quad \int_{-\pi}^{\pi} J_n(t)\, dt = 1.$$

Carefully justifying each step prove that, for any $f \in C(\mathbb{T})$, we have the estimate

$$\|j_n(f) - f\| \leqslant c\,\omega_2(f, \tfrac{1}{n})\,,$$

where $\omega_2(f, t)$ is the second modulus of smoothness of $f$.

Hence, proving the relevant property of $\omega_2(f, t)$, show that if $f$ is twice continuously differentiable, then

$$E_n(f) \leqslant \frac{c_1}{n^2} \|f''\|_{C(\mathbb{T})}\,.$$

**4**

a) State the Kolmogorov criterion for the element of best approximation to a real-valued function $f \in C[0,1]$ from a linear subspace $\mathcal{U}$ of $C[0,1]$.

b) From this criterion, derive the Chebyshev alternation theorem for the element of best approximation to a function $f \in C[0,1]$ from $\mathcal{P}_n$, the space of all algebraic polynomials of degree $\leqslant n$.

c) From first principles, prove that, for any $f \in C[-1,1]$, the polynomial of best approximation is unique. (Just referring to Haar's unicity theorem is not acceptable, but you may use arguments similar to those in its proof.)

**5**

Given a knot sequence $\Delta = (t_i)_{i=1}^{n+k}$, let $\omega_i$ and $\ell_i(\cdot, t)$ be polynomials in $\mathcal{P}_{k-1}$ defined by

1) $\omega_i(x) := (x - t_{i+1}) \cdots (x - t_{i+k-1})$,
2) $\ell_i(\cdot, t)$ interpolates $(\cdot - t)_+^{k-1}$ on $x = t_i, ..., t_{i+k-1}$.

Further, let

$$N_i := (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}$$

be the B-spline of order $k$ with the knots $t_i, \ldots, t_{i+k}$.

a) Prove Lee's formula

$$\omega_i(x) N_i(t) = \ell_{i+1}(x,t) - \ell_i(x,t), \qquad \forall x, t \in \mathbb{R},$$

and derive from it the Marsden identity:

$$(x-t)^{k-1} = \sum_{i=1}^{n} \omega_i(x) N_i(t), \quad t_k < t < t_{n+1}, \quad \forall x \in \mathbb{R}.$$

b) From the Marsden identity, find the coefficients $a_i^{(m)}$ in the B-spline representation of monomials $t^m$:

$$t^m = \sum_{i=1}^{n} a_i^{(m)} N_i(t), \quad t_k < t < t_{n+1}, \quad \text{for} \quad m = 0, \ldots, k-1.$$

**6**

1) Let $\mathbb{X}$ be an inner product space with the scalar product $(\cdot,\cdot)$ and the norm $\|x\| := (x,x)^{1/2}$, and let $\mathcal{U}_n$ be an $n$-dimensional subspace.

a) Prove that $u^* \in \mathcal{U}_n$ is the best approximation to $x \in \mathbb{X}$ from $\mathcal{U}_n$ if and only if

$$(x - u^*, v) = 0 \quad \forall v \in \mathcal{U}_n.$$

b) Let $(u_j)_{j=1}^n$ be a basis for $\mathcal{U}_n$ and let $G = ((u_i, u_j))_{i,j=1}^n$ be the corresponding Gram matrix. Prove that the elements of the Gramian inverse $G^{-1} = (b_{jk})$ are

$$b_{jk} = (\widehat{u}_j, \widehat{u}_k)\,,$$

where $(\widehat{u}_k)$ is the dual basis, i.e., $(u_i, \widehat{u}_k) = \delta_{ik}$. (*Hint.* Use $\delta_{ik} = (G \cdot G^{-1})_{ik}$.)

2) Let $(N_i)$ and $(M_i)$ be the B-spline bases of degree $k-1$ with the $L_\infty$- and $L_1$-normalizations, respectively, defined on a knot sequence $\Delta = (t_i)_{i=1}^{n+k} \subset [0,1]$.

Given $f \in C[0,1]$, let

$$P_{\mathcal{S}}(f) := s^* = \sum_{j=1}^n a_j N_j$$

be the orthogonal projection of $f$ onto $\mathcal{S} := \mathrm{span}\,(N_j)$ with respect to the ordinary inner product $(f,g) = \int_0^1 f(x)g(x)\,dx$. Then $P_{\mathcal{S}}$ is also well defined as an operator from $C[0,1]$ onto $C[0,1]$.

Carefully justifying each step, show that the max-norm of $P_{\mathcal{S}}$ satisfies the inequality

$$\|P_{\mathcal{S}}\|_\infty \leqslant \|G^{-1}\|_{\ell_\infty}$$

where $G = \big\{(M_i, N_j)\big\}_{i,j=1}^h$ is the Gram matrix.

3) From the definition

$$M_i(t) := k[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}$$

prove that the B-splines of degree $k-1$ have finite support. Hence find the bandwidth of the Gram matrix $G = (M_i, N_j)$, i.e., the integer $d$ such that

$$G_{ij} = 0 \quad \text{if} \quad |i-j| \geqslant d$$

**END OF PAPER**