UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2015 1:30 pm to 4:30 pm

PAPER 68

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

*Attempt no more than **THREE** questions from Section A and **ONE** question from Section B.*

*There are **SEVEN** questions in total.*

The questions in Section B carry twice the weight of those in Section A. Questions within each section carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Consider the two-step method

$$\mathbf{y}_{n+2} - \mathbf{y}_n = h[a\mathbf{f}(\mathbf{y}_{n+2}) + 2(1-a)\mathbf{f}(\mathbf{y}_{n+1}) + a\mathbf{f}(\mathbf{y}_n)]$$

for the solution of the ODE $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Here a is a real parameter.

(a) For which values of a is the method convergent?

(b) Determine the order of the method for different values of a.

(c) For which values is the method A-stable?

2

Consider the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \alpha u, \qquad -1 \leqslant x \leqslant 1, \quad t \geqslant 0,$$

given with the initial conditions $u(x,0) = \phi(x)$, $\partial u(x,0)/\partial t = \psi(x)$, $-1 \leqslant x \leqslant 1$, and with zero boundary conditions at $x = \pm 1$. α is a given parameter.

(a) For which values of α is it true that $\lim_{t\to\infty} u(x,t)$ is uniformly bounded for all $x \in [-1,1]$?

(b) The method is solved by the semidiscretized scheme

$$u_m'' = \frac{1}{(\Delta x)^2}(u_{m-1} - 2u_m + u_{m+1}) + \alpha u_m, \qquad m = -N, \ldots, N,$$

where $\Delta x = 1/(N+1)$ and $u_{-N-1}, u_{N+1} \equiv 0$. Determine the values of α for which $\lim_{t\to\infty} u_m(t)$ is uniformly bounded for all $|m| \leqslant N$.

3

(a) Define positive definiteness of a linear operator $\mathcal{L}$, acting in a real Hilbert space $\mathcal{H}$.

(b) Let $\mathcal{L}$ be positive definite in $\mathcal{H}$ and $f \in \mathcal{H}$. Set

$$\mathbf{I}(v) = \langle \mathcal{L}v, v\rangle - 2\langle f, v\rangle, \qquad v \in \mathcal{H}.$$

Prove that $\mathcal{L}u = f$ is the Euler–Lagrange equation corresponding to $\mathbf{I}$ and that the weak solution of $\mathcal{L}u = f$ is the unique minimum of $\mathbf{I}$.

(c) Let $\Omega = [0,1]^2$, $f \in \mathrm{L}_2(\Omega)$ and $a \in \overset{\circ}{\mathrm{H}}{}_2^1(\Omega)$, the Sobolev space which is a closure of $\mathrm{C}^1(\Omega)$ subject to zero Dirichlet boundary conditions. In addition, we stipulate that there exist $0 < a_- < a_+ < \infty$ such that

$$a_- \leqslant a(x,y) \leqslant a_+, \qquad (x,y) \in \Omega.$$

Prove that the *Laplace–Beltrami operator* $\mathcal{L} = -\nabla^\top a \nabla$, acting in Ω with zero Dirichlet boundary conditions is positive definite and determine the variational form $\mathbf{I}$ corresponding to the PDE $\mathcal{L}u = f$.

4

Consider the Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}.$$

(a) Determine the order of this method.

(b) Find the linear stability domain. Is the method A-stable?

(c) Quoting (without proof) all relevant results, determine whether the method is algebraically stable.

5

We consider the parabolic PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a(x)u, \qquad t \geqslant 0, \quad 0 \leqslant x \leqslant 1,$$

given with initial boundary conditions at $t = 0$ and zero Dirichlet boundary conditions at $x = 0$ and $x = 1$. Here a is a continuous function,

$$-\infty < a_0 \leqslant a(x) \leqslant a_+ < \infty, \qquad 0 \leqslant x \leqslant 1.$$

(a) The equation is semidiscretized by simple central differences,

$$u'_m = \frac{1}{(\Delta x)^2}(u_{m-1} - 2u_m + u_{m+1}) + a_m u_m, \qquad m = 1, \ldots, M,$$

where $\Delta x = 1/(M+1)$, $a_m = a(m\Delta x)$ and $u_0, u_{M+1} \equiv 0$. Prove that the method is stable.

(b) The (forward) Euler method is applied to above semidiscretized ODEs and the outcome is

$$u_m^{n+1} = u_m^n + \mu(u_{m-1}^n - 2u_m^n + u_{m+1}^n) + (\Delta t)a_m u_m^n, \qquad m = 1, \ldots, M, \quad n \geqslant 0,$$

where $\mu = \Delta t/(\Delta x)^2$ is the Courant number. Prove that, as long as $\mu \leqslant \frac{1}{2}$, the method is stable.

SECTION B

6

Write an essay on Fourier stability analysis for finite difference methods for PDEs of evolution. You should comment on the suitability and different treatment of different kinds of boundary conditions and accompany your essay with examples.

7

Write an essay on the connection between convergence, order and stability in the numerical solution of ODEs.

END OF PAPER